Myerson's Optimal Auction Design

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We prove Myerson's seminal result,[1] that total expected revenue equals total expected virtual welfare in a DSIC, IR single-parameter auction.

[1] Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981

1 *Mathematical Aside*

Integration is usually performed with respect to the x axis:

$$\int_a^b f(x)\,\mathrm{d}x$$

But sometimes (e.g., in the case of Myerson's formula), it can be convenient to integrate with respect to the y axis instead. To do so, we perform a change of variables: let $y = f(x)$, so that $x = f^{-1}(y)$.[2] This change yields:

$$\int_a^b f(x)\,\mathrm{d}x = \int_{f(a)}^{f(b)} f^{-1}(y)\,\mathrm{d}y.$$

[2] assuming f^{-1} exists: e.g., if f is continuous and (strictly) increasing. If f is only weakly increasing, the same argument goes through, albeit only by invoking some more heavy-duty machinery (i.e., generalized inverses and Stieltjes integrals).

When integrating with respect to x, we imagine summing all of the vertical rectangles from a to b with width x and height $f(x)$. Similarly, when integrating with respect to y, we imagine summing all of the horizontal rectangles from $f(a)$ to $f(b)$ with height y and width $x = f^{-1}(y)$.

Applying the same change of variables again (i.e., $y = f(x)$), we can again express this latter summation as integrating along the x-axis, but this time, with respect to $f(x)$, because this assignment yields $\mathrm{d}y = \mathrm{d}f(x)$:

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,\mathrm{d}y = \int_{f^{-1}(f(a))}^{f^{-1}(f(b))} f^{-1}(f(x))\,\mathrm{d}f(x) = \int_a^b x\,\mathrm{d}f(x).$$

2 *Myerson's Lemma: Recap*

Recall Myerson's payment characterization formula, assuming $p_i(\underline{v}_i, \mathbf{v}_{-i}) - \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) = c$, namely:

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i})\,\mathrm{d}z + c. \tag{1}$$

In our derivation of Myerson's optimal auction, we rely on an equivalent payment characterization formula, in which we integrate the allocation function $x_i(z, \mathbf{v}_{-i})$ with respect to $\mathrm{d}z$, as follows:

$$p_i(v_i, \mathbf{v}_{-i}) = \int_{\underline{v}_i}^{v_i} z\,\mathrm{d}x_i(z, \mathbf{v}_{-i}) + c. \tag{2}$$

3 Optimal Auction Design

In this lecture, using Myerson's lemma as a starting point, we will prove Myerson's theorem, which dictates the design of an optimal (i.e., revenue-maximizing) auction. This theorem relates payments (i.e., revenue) to **virtual welfare**, which, as we will see, is defined in terms of each bidder's **virtual value** function φ_i.

Like Myerson's lemma, the theorem concerns a single-parameter auction. Furthermore, each bidder i's values v_i are drawn from a continuous strictly increasing distribution F_i, with support $T_i = [\underline{v}_i, \overline{v}_i]$, for some lowest and highest types $\underline{v}_i, \overline{v}_i \in \mathbb{R}_+$.

Theorem 3.1 (Myerson). *In a DSIC,[3] IR[4] single-parameter auction, bidder i's expected payment is equal to bidder i's expected virtual welfare: i.e.,*

$$\mathop{\mathbb{E}}_{v_i \sim F_i} [p_i(v_i, \mathbf{v}_{-i})] = \mathop{\mathbb{E}}_{v_i \sim F_i} [\varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})] , \tag{3}$$

assuming $v_i \sim F_i$; $f_i(v_i) > 0$, for all $v_i \in T_i$; and $T_i = [\underline{v}_i, \overline{v}_i]$, for $-\infty < \underline{v}_i < \overline{v}_i < \infty$.

[3] In fact, BIC, which is weaker than DSIC, suffices. Either assumption can be used with the corresponding payment formula.

[4] We assume the lowest type is allocated nothing and pays nothing, so that $\underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i}) = 0$.

Proof. For a fixed $\mathbf{v}_{-i} \in T_{-i}$, bidder i's expected payment is as follows:

$$\begin{aligned}
\mathop{\mathbb{E}}_{v \sim F_i} [p_i(v, \mathbf{v}_{-i})] &= \int_{\underline{v}_i}^{\overline{v}_i} p_i(v, \mathbf{v}_{-i}) f_i(v) \, \mathrm{d}v \\
&= \int_{\underline{v}_i}^{\overline{v}_i} \left[\int_{\underline{v}_i}^{v} z \, \mathrm{d}x_i(z, \mathbf{v}_{-i}) \right] f_i(v) \, \mathrm{d}v \\
&= \int_{\underline{v}_i}^{\overline{v}_i} \left[\int_{\underline{v}_i}^{v} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) \mathrm{d}z \right] f_i(v) \, \mathrm{d}v \\
&= \int_{\underline{v}_i}^{\overline{v}_i} \left[\int_{\underline{v}_i}^{v} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) f_i(v) \right] \mathrm{d}z \, \mathrm{d}v.
\end{aligned}$$

Now, note that the allocation function is non-negative, that $f_i(v)$ is positive by assumption, and that the type space is bounded s.t. the bid z is also non-negative. Consequently, $z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) f_i(v)$ is a non-negative function. By Tonelli's theorem the order of the integration can therefore be reversed.

To reverse the order of integration we must also change the limits of integration. Notice that v ranges from $\underline{v}_i$ to $\overline{v}_i$, and that, for any fixed $v = V$, z ranges from $\underline{v}_i$ to V. If we instead let z range from $\underline{v}_i$ to $\overline{v}_i$, then for any fixed $z = Z$, v ranges from Z to $\overline{v}_i$. We attempt to depict this argument (Tonelli's theorem) in Figure 1.

Using our knowledge of the region of integration, we can switch the order of integration as follows:

$$\int_{\underline{v}_i}^{\overline{v}_i} \int_{\underline{v}_i}^{v} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) f_i(v) \, \mathrm{d}z \, \mathrm{d}v$$

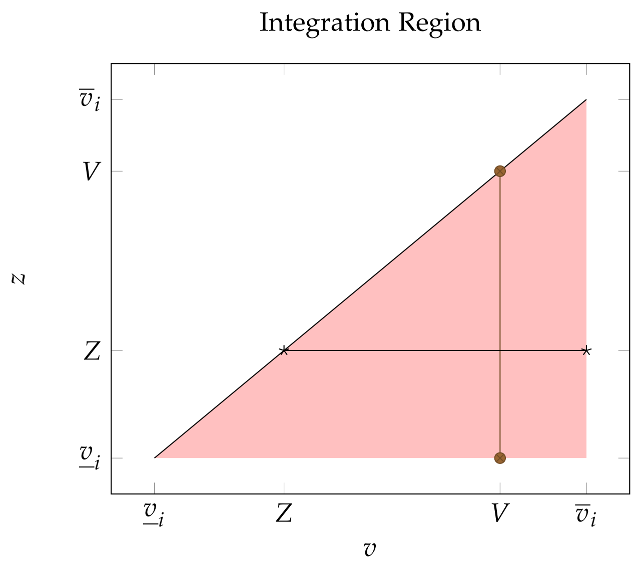

Figure 1: Tonelli's theorem: The shaded area represents the values of v and z used to evaluate bidder i's contribution to expected revenue.

$$
\begin{aligned}
&= \int_{\underline{v}_i}^{\overline{v}_i} \int_{z}^{\overline{v}_i} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) f_i(v)\,\mathrm{d}v\,\mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) \int_{z}^{\overline{v}_i} f_i(v)\,\mathrm{d}v\,\mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) F_i(v)\Big|_{z}^{\overline{v}_i} \mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) (1 - F_i(z))\ \mathrm{d}z.
\end{aligned}
$$

Next, we use integration by parts:

$$
\int_a^b u\,\mathrm{d}v = uv\big|_a^b - \int_a^b v\,\mathrm{d}u,
$$

where we let

$$
\begin{aligned}
u &= z\,[1 - F_i(z)] & \mathrm{d}u &= [-zf_i(z) + 1 - F_i(z)]\ \mathrm{d}z \\
\mathrm{d}v &= \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) \mathrm{d}z & v &= x_i(z, \mathbf{v}_{-i}),
\end{aligned}
$$

to get:

$$
\begin{aligned}
&\int_{\underline{v}_i}^{\overline{v}_i} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) (1 - F_i(z))\ \mathrm{d}z \\
&= \underbrace{z\,(1 - F_i(z))}_{u}\, \underbrace{x_i(z, \mathbf{v}_{-i})}_{v} \Big|_{\underline{v}_i}^{\overline{v}_i} - \int_{\underline{v}_i}^{\overline{v}_i} \underbrace{x_i(z, \mathbf{v}_{-i})}_{v}\, \underbrace{(-zf_i(z) + 1 - F_i(z))\ \mathrm{d}z}_{du} \\
&= (0 - 0) + \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i})\,(zf_i(z) - (1 - F_i(z)))\ \mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i})\,(zf_i(z) - (1 - F_i(z)))\ \mathrm{d}z.
\end{aligned}
$$

The first 0 is $1 - F_i(\overline{v}_i) = 1 - 1$. The second 0 follows from the assumption that lowest types are never allocated.

What we have at this point is not quite an expectation. But, since $f_i(z) > 0$ for all $z \in T_i$, multiplying by $f_i(z)/f_i(z)$ does no harm:

$$1 - F_i(z) = \left(\frac{1 - F_i(z)}{f_i(z)}\right) f_i(z).$$

After this manuever, our expression simplifies as follows:

$$\begin{aligned}
&\int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i}) \left(z f_i(z) - (1 - F_i(z))\right) \mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i}) \left(z f_i(z) - \left(\frac{1 - F_i(z)}{f_i(z)}\right) f_i(z)\right) \mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i}) \left(z - \frac{1 - F_i(z)}{f_i(z)}\right) f_i(z) \, \mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i}) \varphi_i(z) f_i(z) \, \mathrm{d}z \\
&= \mathop{\mathbb{E}}_{z \sim F_i} \left[\varphi_i(z) x_i(z, \mathbf{v}_{-i})\right],
\end{aligned}$$

where $\varphi_i(z) = z - \frac{1 - F_i(z)}{f_i(z)}$. We call this quantity **virtual value**. Correspondingly, we call the quantity $\varphi_i(v) x_i(v, \mathbf{v}_{-i})$ **virtual welfare**, since it is allocation times virtual value, instead of allocation times value.

Finally, renaming the bound variable z as the usual v yields:

$$\mathop{\mathbb{E}}_{v \sim F_i} \left[p_i(v, \mathbf{v}_{-i})\right] = \mathop{\mathbb{E}}_{v \sim F_i} \left[\varphi_i(v) x_i(v, \mathbf{v}_{-i})\right]$$

□

Now that we can relate a single bidder i's contribution to total expected revenue to its contribution to total expected virtual welfare, it is straightforward to show that *total* expected revenue is equal to *total* expected virtual welfare.

Corollary 3.2 (Myerson)**.** *In a DSIC, IR single-parameter auction, total expected revenue is equal to total expected virtual welfare: i.e.,*

$$\sum_{i \in [n]} \mathop{\mathbb{E}}_{\mathbf{v} \sim F} \left[p_i(v_i, \mathbf{v}_{-i})\right] = \sum_{i \in [n]} \mathop{\mathbb{E}}_{\mathbf{v} \sim F} \left[\varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})\right], \tag{4}$$

assuming, for all players $i \in [n]$, $v_i \sim F_i$; $f_i(v_i) > 0$, for all $v_i \in T_i$; and $T_i = [\underline{v}_i, \overline{v}_i]$, for $-\infty < \underline{v}_i < \overline{v}_i < \infty$.

Proof. Summing over all bidders,

$$\sum_{i \in [n]} \mathop{\mathbb{E}}_{v \sim F_i} \left[p_i(v, \mathbf{v}_{-i})\right] = \sum_{i \in [n]} \mathop{\mathbb{E}}_{v \sim F_i} \left[\varphi_i(v) x_i(v, \mathbf{v}_{-i})\right].$$

Take expectations with respect to $\mathbf{v}_{-i}$ to get

$$\mathop{\mathbb{E}}_{\mathbf{v}_{-i}\sim F_{-i}}\left[\sum_{i\in[n]}\mathop{\mathbb{E}}_{v\sim F_i}\left[p_i(v,\mathbf{v}_{-i})\right]\right] = \mathop{\mathbb{E}}_{\mathbf{v}_{-i}\sim F_{-i}}\left[\sum_{i\in[n]}\mathop{\mathbb{E}}_{v\sim F_i}\left[\varphi_i(v)x_i(v,\mathbf{v}_{-i})\right]\right].$$

Finally, by linearity of expectations, we complete the proof:

$$\sum_{i\in[n]}\mathop{\mathbb{E}}_{\mathbf{v}\sim F}\left[p_i(\mathbf{v})\right] = \sum_{i\in[n]}\mathop{\mathbb{E}}_{\mathbf{v}\sim F}\left[\varphi_i(v_i)x_i(\mathbf{v})\right].$$

□

Myerson's theorem tells us that we can maximize revenue by maximizing virtual welfare. In other words, Myerson's theorem reframes revenue-maximization (i.e., an optimization problem) as the implementation problem of finding outcomes of maximal virtual welfare.

So how do we maximize virtual welfare? Well, only bidders with the highest virtual values should be allocated; that is for sure. But more than that, bidders should only be allocated if their virtual values are non-negative; otherwise, virtual welfare would decrease!

Myerson's lemma (Lecture 4) further tells us that incentive compatibility mandates a monotone allocation rule: i.e., as values (weakly) increase, so, too, should the allocation function. But as allocation in the optimal auction is according to virtual values not values themselves, ensuring (weak) monotonicity requires a further assumption, namely that virtual values are (weakly) increasing in values.

A distribution is called **regular** if the corresponding virtual value function is weakly increasing in values. Without this assumption, a higher value could result in a lower virtual value, and a correspondingly lower allocation probability. The regularity assumption ensures that allocating in order of virtual values is monotone, a requirement if the optimal auction is to be incentive compatible.

References

[1] Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.