Regular and MHR Distributions

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We define the hazard rate function, and then regular and monotone hazard rate (MHR) distributions, and we contrast MHR distributions with regular distributions.

1 *Background: Derivative Fundamentals*

Recall the **difference quotient** definition of a derivative $g'(x)$ of a function g at a point x:

$$g'(x) = \lim_{\delta \to 0} \frac{g(x+\delta) - g(x)}{\delta}.$$

This expression describes the limit of the average rate of change of f as the interval δ between two points shrinks to 0. When applied to a CDF $F(x) = \Pr[X \leq x]$, this definition yields the following:

$$\begin{aligned} f(x) &= \lim_{\delta \to 0} \frac{\Pr[X \leq x+\delta] - \Pr[X \leq x]}{\delta} \\ &= \lim_{\delta \to 0} \frac{\Pr[x < X \leq x+\delta]}{\delta}. \end{aligned}$$

The numerator is the probability of the event, "the random variable X realizes a value in the interval $(x, x+\delta)$," while the denominator δ is the width of the interval. Hence, this fraction describes the average probability mass, per unit of X. Taking the limit as $\delta \to 0$ yields the instantaneous rate of change in probability, i.e., the PDF $f(x)$:

$$f(x) = \lim_{\delta \to 0} \frac{\Pr[x < X \leq x+\delta]}{\delta}.$$

2 *The Hazard Rate*

Assume T is a continuous random variable representing whether an event (e.g., marriage, parenthood, migration, death, etc.) has occurred by time $t \geq 0$ with PDF $f(t) > 0$ and CDF $F(t)$: i.e.,

$$F(t) = \Pr(T \leq t) = \int_0^t f(x)dx.$$

The **survival distribution** $S(t)$ indicates the probability that the event has *not* occurred by time t:

$$S(t) = 1 - F(t) = \Pr(T > t) = \int_t^\infty f(x)dx.$$

The **hazard rate**[1] function $h(t)$ describes the instantaneous rate of occurrence of the event, *given that it has not occurred by time t*:

[1] also called a failure rate

$$h(t) = \lim_{\delta \to 0} \frac{\Pr[t < T \leq t + \delta \mid T > t]}{\delta}.$$

The numerator in this expression is the probability that the event will occur in the interval $(t, t+\delta]$, *given that it has not occurred by time t*, and the denominator, as usual, is the width of the interval. Hence, this fraction describes the rate of occurrence of the event, per unit of time. Taking the limit as $\delta \to 0$ yields the instantaneous rate.

The hazard rate function can be simplified as follows:

$$\begin{aligned}
h(t) &= \lim_{\delta \to 0} \frac{\Pr[t < T \leq t + \delta \mid T > t]}{\delta} \\
&= \lim_{\delta \to 0} \left(\frac{\Pr[t < T \leq t + \delta, T > t]}{\delta} \right) \left(\frac{1}{\Pr[T > t]} \right) \\
&= \lim_{\delta \to 0} \left(\frac{\Pr[t < T \leq t + \delta]}{\delta} \right) \left(\frac{1}{\Pr[T > t]} \right) \\
&= \frac{f(t)}{S(t)} \\
&= \frac{f(t)}{1 - F(t)}.
\end{aligned}$$

This latter form often rears its[2] head in auction analyses.

[2] no-longer-seeming-so-ugly

Remark 2.1. Since for an arbitrary real-valued function g,

$$\frac{d}{dx} \ln g(x) = \frac{\frac{d}{dx} g(x)}{g(x)},$$

and since $\frac{d}{dt} S(t) = -f(t)$, it follows that

$$h(t) = \frac{f(t)}{S(t)} = \frac{-\frac{d}{dt} S(t)}{S(t)} = -\frac{d}{dt} \ln S(t).$$

Usually, a hazard rate is assumed to be increasing, decreasing, or constant with time. An increasing hazard rate signifies that the unit is becoming more and more prone to failure. A decreasing hazard rate means the opposite: the unit is improving with time. Other possibilities include a U-shaped, or an upside-down U-shaped, hazard rate. The former is often used to model a human life span, because early in life we are very vulnerable, while at mid-life risks level off, until later in life when we become vulnerable again.

3 *Regular and MHR Distributions*

Observe that the virtual value function of a distribution F relates to the *inverse* hazard rate function h as follows:

$$\varphi(v) = v - \frac{1 - F(v)}{f(v)} = v - \frac{1}{h(v)}.$$

Now recall that Myerson's optimal auction design recipe requires that the virtual value function be weakly increasing in values: i.e., for $v \geq t \in T$, $\varphi(v) \geq \varphi(t)$. Distributions for which the corresponding virtual value function satisfies this property are called **regular**.

A related, and stronger, condition is that the hazard rate function be weakly increasing: i.e., for $v \geq t \in T$, $h(v) \geq h(t)$. This condition is called the **monotone hazard rate** (MHR) condition. Many common distributions satisfy the MHR condition: e.g., the uniform, the normal, and the exponential distributions.

Remark 3.1. The MHR condition can be interpreted to mean that a distribution is *not* "heavy tailed." For heavy-tailed distributions, tail events (i.e., extreme values) are more likely than under the normal distribution. Whereas observing a 6σ-event of a normal distributed random variable is nearly impossible, it might be merely uncommon for a random variable with a heavy-tailed distribution. Technically speaking, a random variable is heavy-tailed if its survival function decays more slowly than the survival function of an exponentially-distributed random variable.

The exponential distribution, call it f, models time between events. More specifically, for an exponentially-distributed random variable X with rate parameter $\lambda > 0$,

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In other words, the probability the event occurs by time t is $1 - e^{-\lambda t}$. Moreover, the hazard rate, or the instantaneous rate of occurrence, *given survival up to time t*, is

$$h(t) = \frac{f(t)}{S(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda,$$

which is constant; it does not depend on how much time has lapsed.

The survival distribution corresponding to the exponential is $S(t) = \Pr(T > t) = e^{-\lambda t}$, meaning the survival probability drops off multiplicatively.[3] In contrast, the Pareto-distribution is heavy-tailed: its survival distribution, $S(t) = \Pr(T > t) \sim t^{-\alpha}$, for some $\alpha > 0$, decays much more slowly that the exponential's.

[3] Additive decay means subtracting a constant with each unit of time, while multiplicative decay means dividing by a constant with each unit of time.

We observe that MHR distributions are regular.

Proposition 3.2. *MHR implies regularity.*

Proof. If the hazard rate $h(v)$ is weakly increasing in values, then the inverse hazard rate $1/h(v)$ is weakly decreasing in values: for $v \geq t$,

$$\begin{aligned} h(v) \geq h(t) & \qquad \Longleftrightarrow \\ \frac{1}{h(v)} \leq \frac{1}{h(t)} & \qquad \Longleftrightarrow \\ \frac{1-F(v)}{f(v)} \leq \frac{1-F(t)}{f(t)}. & \end{aligned}$$

Likewise,

$$-\frac{1-F(v)}{f(v)} \geq -\frac{1-F(t)}{f(t)}.$$

The virtual value function is thus weakly increasing in values:

$$v - \frac{1-F(v)}{f(v)} \geq v - \frac{1-F(t)}{f(t)} \geq t - \frac{1-F(t)}{f(t)}.$$

Therefore, MHR implies regularity. □

Although MHR implies regularity, the two are not equivalent (example forthcoming). One way to see that the MHR condition is stronger than regularity is the following: whereas regularity means the virtual value function is weakly increasing, MHR implies the virtual value function is *strictly* increasing: for $\delta > 0$,

$$\begin{aligned} h(v+\delta) \geq h(v) & \qquad \text{MHR} \\ -\frac{1}{h(v+\delta)} \geq -\frac{1}{h(v)} & \qquad \Longleftrightarrow \\ v - \frac{1}{h(v+\delta)} \geq v - \frac{1}{h(v)} & \qquad \Longrightarrow, \text{ since } \delta > 0, \\ v + \delta - \frac{1}{h(v+\delta)} > v - \frac{1}{h(v)} & \qquad \Longleftrightarrow \\ \varphi(v+\delta) > \varphi(v) & \qquad \text{virtual values are strictly increasing} \end{aligned}$$

Finally, to show that the sets of MHR and regular distributions are distinct, we present an example of a distribution that satisfies regularity, but not MHR. This distribution indeed has heavy tails.

Example 3.3 (Regular, and not MHR)**.** The distribution

$$F(v) = 1 - \frac{1}{v+1}$$

has density

$$f(v) = \frac{1}{(v+1)^2},$$

and support $[0, \infty)$. The PDF and CDF are shown in Figure 1.

The hazard rate function is

$$\begin{aligned} h(v) &= \frac{f(v)}{1 - F(v)} \\ &= \frac{\frac{1}{(v+1)^2}}{1 - \left(1 - \frac{1}{v+1}\right)} \\ &= \frac{1}{v+1}. \end{aligned}$$

Since $h(v)$ is strictly decreasing in v, F does not satisfy the MHR condition. The virtual value function, however, is constant, and thus a weakly increasing function of values:

$$\begin{aligned} \varphi(v) &= v - \frac{1}{h(v)} \\ &= v - (v+1) \\ &= -1. \end{aligned}$$

Therefore, F satisfies the regularity, but not the MHR, condition.

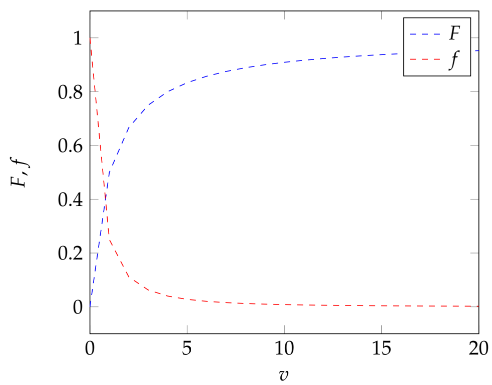

Figure 1: The PDF and CDF of a heavy-tailed distribution. At $v = 20$, $F(20) = .95$. At $v = 100$, $F(100) = .99$. At $v = 1000$, $F(1000) = .999$. This is an instance of the Burr distribution, $F(v, c, k) = 1 - (1 + v^c)^{-k}$, where $c = k = 1$.

Although F is regular, Myerson's scheme for allocating only to bidders with non-negative virtual values would not maximize revenue in an auction where values are distributed according to F, as no good would ever be allocated, so revenue would always be 0. In contrast, running a first- or second-price auction would generate positive expected revenue. Myerson's auction is not optimal for this distribution because its support is unbounded, so Tonelli's theorem (which we invoked to swap the order of integration in our proof of Myerson's theorem) does not apply.