Applications of Myerson's Lemma

CSCI 1440/2440

2025-09-24

We apply Myerson's lemma to solve the single-good auction, and the generalization in which there are k homogeneous goods: i.e., k identical copies of the good. Our objective is welfare maximization.

1 *Welfare-Maximizing Auctions*

We can interpret Myerson's lemma as providing a recipe for designing an dominant-strategy incentive compatible (DSIC) and individually rational (IR) welfare-maximizing auction. The first step is to construct an (computationally) efficient feasible allocation rule that is monotonic in bidders' values, and the second, is to plug that rule into Myerson's payment formula to guarantee the incentive properties. When the allocation rule also achieves economic efficiency—meaning it optimizes (or approximately optimizes) welfare—we say that the auction is solved (or approximately solved).

2 *Single-Good Auction*

Our first application of Myerson's lemma is a simple sanity check. We have already discussed a DSIC auction design for the single-parameter setting with one good: the second-price auction, in which the highest bidder wins and pays the second-highest bid. Here, we comfirm that Myerson's lemma leads us to the same conclusion.

Assume a single good auction with n, bidders, each with a private value v_i in the range $T_i = [0, \overline{v}_i]$. (For simplicity, we assume $\underline{v}_i = 0$, for all bidders $i \in [n]$.)

Welfare Maximization Recall that welfare is the quantity $\sum_i v_i x_i(\mathbf{v})$, assuming ex-post feasibility: i.e., $\mathbf{x} \in \{0,1\}^n$ and $||\mathbf{x}|| \leq 1$. This quantity is maximized by awarding the good to a bidder with the highest value: i.e., a bidder i^* s.t.

$$i^* \in \arg\max_i v_i,$$

Monotonicity Fix a bidder i and a profile $\mathbf{v}_{-i}$. The necessary and sufficient condition for i to be allocated is that she bid higher than the highest bid among bidders other than i. This bid, which we denote b^*, is called the **critical bid**: $b^* \equiv \max_{j \neq i} v_j =$ 2nd-highest bid.

This allocation rule, which is monotonically weakly increasing in bidder i's values, is plotted in Figure 1.

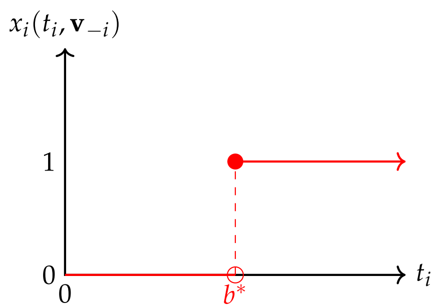

Figure 1: Bidder i's allocation rule, for a fixed $\mathbf{v}_{-i}$. This rule is monotone.

Claim This allocation rule is monotonically weakly increasing.

Proof. If $v_i < b^*$, then $x_i(v_i, \mathbf{v}_{-i})$ is 0, so increasing i's bid cannot possibly lower i's allocation. On the other hand, if $v_i \geq b^*$ is a winning bid, so that $x_i(v_i, \mathbf{v}_{-i})$ is 1, then $x_i(v_i + \epsilon, \mathbf{v}_{-i})$ still equals 1, for all $\epsilon > 0$. In sum, for all $v_i \in T_i$ and for all $\epsilon > 0$, $x_i(v_i + \epsilon, \mathbf{v}_{-i}) \geq x_i(v_i, \mathbf{v}_{-i})$: i.e., x_i is monotonically weakly increasing in values. □

Payments Assuming bidder i is a winner, her payment is as follows:[1]

$$\begin{aligned} p_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, dz, \\ &= v_i \cdot 1 - \left[\int_0^{b^*} 0\, dz + \int_{b^*}^{v_i} 1\, dz \right] \\ &= v_i - (v_i - b^*) \\ &= b^*. \end{aligned}$$

[1] under our running assumption that lowest types are not allocated

We split up the integral in this way because the allocation for bidding less than b^* is 0, while the allocation for bidding more is 1. Indeed, the payment b^* is the second-highest price. This payment is the region shaded in red in Figure 2.

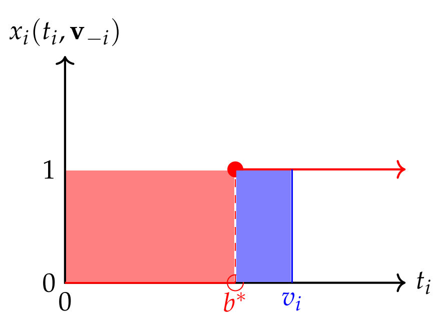

Figure 2: Bidder i's payment function, for a fixed $\mathbf{v}_{-i}$. Payment is shown in red, and utility, for v_i as shown, in blue.

We conclude that the combination of an allocating to a highest bidder together with charging (only) the winner of the auction the second-highest bid is IC and IR. Since this allocation rule is both economically and compuationally efficient, this auction is solved.

3 *k-Good Auction*

Now consider an auction with $k \geq 1$ identical copies of a good and $n \geq k$ bidders, each with a private value v_i *for a single copy of the good* in the range $T_i = [0, \overline{v}_i]$.

Welfare Maximization Problem Generalizing the single-good case, welfare is the quantity $\sum_i v_i x_i(\mathbf{v})$, assuming ex-post feasibility: i.e., $\mathbf{x} \in \{0,1\}^n$ and $||\mathbf{x}|| \leq k$. This quantity is maximized by awarding the goods to the k highest bidders: i.e., by setting those entries of $\mathbf{x}$ that correspond to the k largest bids to 1, and all others to 0.

Monotonicity Fix a bidder i and a profile $\mathbf{v}_{-i}$. The necessary and sufficient condition for i to be allocated is that her bid be among the k highest bids. The highest bid among those bidders who are not among the top k-highest bidders, which we denote b^*, is called the **critical bid**: $b^* \equiv k\text{th-highest}_{j \neq i} v_j = k + 1\text{st-highest bid}$.

Since the condition for being allocated is the same as it was in the single-good case (i.e., simply bidding above the critical bid), the allocation rule is the same as it was in the single-good case. This allocation rule, which is monotonically weakly increasing in bidder i's values, is plotted in Figure 3.

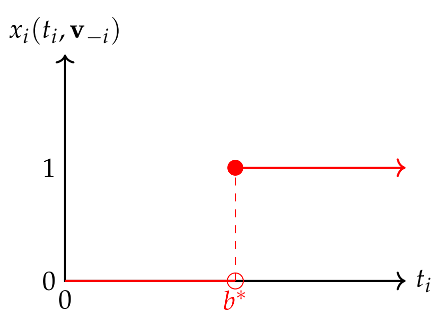

Figure 3: Bidder i's allocation rule, for a fixed $\mathbf{v}_{-i}$. This rule is monotone.

Claim This allocation rule is monotonically weakly increasing.

Proof. If $v_i < b^*$, then $x_i(v_i, \mathbf{v}_{-i})$ is 0, so increasing i's bid cannot possibly lower i's allocation. On the other hand, if $v_i \geq b^*$ is a winning bid, so that $x_i(v_i, \mathbf{v}_{-i})$ is 1, then $x_i(v_i + \epsilon, \mathbf{v}_{-i})$ still equals 1, for all $\epsilon > 0$. In sum, for all $v_i \in T_i$ and for all $\epsilon > 0$,

$x_i(v_i + \epsilon, \mathbf{v}_{-i}) \geq x_i(v_i, \mathbf{v}_{-i})$: i.e., x_i is monotonically weakly increasing in values. □

Payments Assuming bidder i is a winner, her payment is as follows:[2]

[2] under our running assumption that lowest types are not allocated

$$\begin{aligned} p_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\,\mathrm{d}z, \\ &= v_i \cdot 1 - \left[\int_0^{b^*} 0\,\mathrm{d}z + \int_{b^*}^{v_i} 1\,\mathrm{d}z\right] \\ &= v_i - (v_i - b^*) \\ &= b^*. \end{aligned}$$

Since the condition for being allocated is the same as it was in the single-good case—simply bidding higher than b^*—this payment calculation is the same as it was in the single-good case. The payment b^* is the $k+1$st-highest price; hence, this auction for k goods generalizes the second-price auction for a single good. This payment is the region shaded in red in Figure 4.

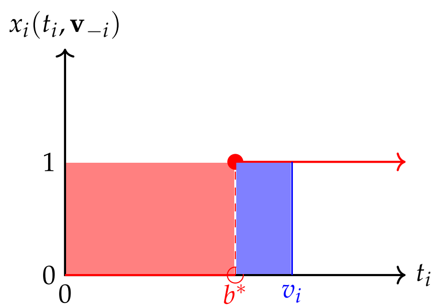

Figure 4: Bidder i's payment function, for a fixed $\mathbf{v}_{-i}$. Payment is shown in red, and utility, for v_i as shown, in blue.

We conclude that the combination of allocating to the k highest bidders together with charging (only) the winners of the auction the kth-highest bid is IC and IR. Since this allocation rule is both economically and compuationally efficient (simply sort the bids, and allocate to the top k), this auction is solved. This solution is called the k-Vickrey auction.

A two-good example. Imagine three bidders, b_1, b_2 and b_3, and two goods. The bidders' type spaces are closed intervals, but with different bounds: bidder i's value $v_i \in [0, i]$.

Let v_i represent bidder i's realized value. Suppose $v_1 = 5/6$, $v_2 = 2$, and $v_3 = 7/4$. What happens in this example in the welfare-maximizing auction, IC, IR, and ex-post feasible auction?

To answer this question, we do the following:

1. Sort the bidders' values.

2. Find the winners: i.e., the bidders with the two highest values.

3. Determine the critical bid, and hence the winners' payments.

Table 1: Example Two-Good Auction

i	v_i	Rank	Winner?	Critical bid	Payment
1	$5/6$	3	no	n/a	n/a
2	2	1	yes	5/6	5/6
3	$7/4$	2	yes	5/6	5/6

These steps are illustrated in Table 1. Bidders 2 and 3 are allocated the goods, as they have the two highest values. They each pay the critical bid, which in this two-good auction is the third-highest value.