Non-regular Distributions & Ironing

CSCI 1440/2440

2025-10-15

We describe **ironing**, a treatment for non-regular distributions, which is applicable to Myerson's optimal auction, among others.

1 *Ironing*

A regular distribution F implies a monotone weakly increasing virtual value function φ, and a concave revenue curve R. If F is not regular, however, the virtual value function may not be weakly increasing, and likewise, the revenue curve may not be concave. In such cases, maximizing virtual welfare may not yield a monotonic allocation rule, which would violate incentive compatibility. In this lecture, we describe how to handle non-regular distributions.

Given a possibly non-regular distribution F, we define an **ironed revenue curve** as the smallest concave function $\overline{R}$ that upper bounds the revenue curve R built from F. Correspondingly, we define the **ironed virtual value function** as the derivative of $\overline{R}$: i.e.,

$$\overline{\varphi}(v(q)) = \frac{\mathrm{d}\overline{R}(q)}{\mathrm{d}q}.$$

Since $\overline{R}$ is concave, it follows that $\overline{\varphi}$ is weakly increasing (in value space). Consequently, we can maximize ironed virtual welfare, and thereby arrive at a monotone weakly increasing allocation rule.

Our primary goal in this lecture is to prove that since $\overline{R}$ upper bounds R, expected ironed virtual welfare $\overline{\varphi}$ upper bounds expected virtual welfare φ, assuming a monotonic allocation rule $\hat{x}$, i.e.,

$$\mathop{\mathbb{E}}_{v \sim F}\left[\varphi(v)\hat{x}(v)\right] \leq \mathop{\mathbb{E}}_{v \sim F}\left[\overline{\varphi}(v)\hat{x}(v)\right].$$

Equivalently, letting $\hat{y}(q(v)) = \hat{x}(v(q))$,

$$\mathop{\mathbb{E}}_{q \sim U[0,1]}\left[\varphi(q)\hat{y}(q)\right] \leq \mathop{\mathbb{E}}_{q \sim U[0,1]}\left[\overline{\varphi}(q)\hat{y}(q)\right].$$

These two values, namely expected virtual welfare and expected ironed virtual welfare, are actually equal in all concave regions; they can only differ in non-concave regions, as a result of concavification. The proof concludes by showing that these two values do not in fact differ even in the non-concave regions after concavification. Therefore, ironing virtual value functions, which constrains the allocation rule to be monotone, does not actually decrease expected virtual welfare (i.e., expected revenue) at all!

We establish these results for interim allocation rules of the form $\hat{x}(v) = \mathbb{E}_{\mathbf{v}_{-i}\sim F_{-i}}[x(v, \mathbf{v}_{-i})]$, but the same technique also applies to pointwise allocation rules, albeit with more notation.

Our proof now comprises three steps:

1. First, we show that total expected virtual welfare can be expressed as follows:

$$\mathop{\mathbb{E}}_{v\sim F}[\varphi(v)\hat{x}(v)] = \mathop{\mathbb{E}}_{q\sim U(0,1)}\left[\left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\hat{y}(q)\right].$$

2. Second, we show that this alternative expression for total expected virtual welfare can also be expressed as follows:

$$\mathop{\mathbb{E}}_{q\sim U(0,1)}\left[\left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\hat{y}(q)\right] = \mathop{\mathbb{E}}_{q\sim U(0,1)}\left[-R(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right].$$

3. Finally, we prove that if $\hat{y}(q)$ is weakly decreasing with respect to q, then

$$\mathop{\mathbb{E}}_{q\sim U(0,1)}\left[-R(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right] \leq \mathop{\mathbb{E}}_{q\sim U(0,1)}\left[-\overline{R}(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right].$$

4. Putting it all together, these three claims establish:

$$\begin{aligned}
\mathop{\mathbb{E}}_{v\sim F}[\varphi(v)\hat{x}(v)] &= \mathop{\mathbb{E}}_{q\sim U(0,1)}\left[\left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\hat{y}(q)\right] \\
&= \mathop{\mathbb{E}}_{q\sim U(0,1)}\left[-R(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right] \\
&\leq \mathop{\mathbb{E}}_{q\sim U(0,1)}\left[-\overline{R}(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right] \\
&= \mathop{\mathbb{E}}_{q\sim U(0,1)}\left[\left(\frac{\mathrm{d}\overline{R}(q)}{\mathrm{d}q}\right)\hat{y}(q)\right] \\
&= \mathop{\mathbb{E}}_{v\sim F}[\overline{\varphi}(v)\hat{x}(v)].
\end{aligned}$$

1.1 Part I

The virtual value function is the derivative of the revenue curve function:

$$\varphi(v(q)) = \frac{\mathrm{d}R(q)}{\mathrm{d}q}.$$

Make the following substitutions:

$$\begin{aligned}
q &= 1 - F(v) \\
\mathrm{d}q &= -f(v)\,\mathrm{d}v \\
q(\underline{v}) &= 1
\end{aligned}$$

$$q(\overline{v}) = 0.$$

Making these substitutions yields:

$$\begin{aligned}
\mathop{\mathbb{E}}_{v\sim F}[\varphi(v)\hat{x}(v)] &= \int_{\underline{v}}^{\overline{v}} \varphi(v)\hat{x}(v)f(v)\,\mathrm{d}v \\
&= \int_1^0 \left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\hat{y}(q)(-1)\,\mathrm{d}q \\
&= \int_0^1 \left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\hat{y}(q)\,\mathrm{d}q \\
&= \mathop{\mathbb{E}}_{q\sim U(0,1)}\left[\left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\hat{y}(q)\right].
\end{aligned}$$

1.2 Part II

We start by expanding the expectation:

$$\mathop{\mathbb{E}}_{q\sim U(0,1)}\left[\left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\hat{y}(q)\right] = \int_0^1 \left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\hat{y}(q)\,\mathrm{d}q.$$

We then use integration by parts, with the following substitutions:

$$\begin{aligned}
u &= \hat{y}(q) \\
\mathrm{d}u &= \left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\mathrm{d}q \\
\mathrm{d}v &= \left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\mathrm{d}q \\
v &= R(q).
\end{aligned}$$

Integration by parts yields:

$$\begin{aligned}
\int_0^1 \left(\frac{\mathrm{d}R(q)}{\mathrm{d}q}\right)\hat{y}(q)\,\mathrm{d}q &= R(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\mathrm{d}q\Big|_0^1 - \int_0^1 R(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\mathrm{d}q \\
&= -\int_0^1 R(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\mathrm{d}q \\
&= \mathop{\mathbb{E}}_{q\sim U(0,1)}\left[-R(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right].
\end{aligned}$$

1.3 Part III

Since $\overline{R}$ is the smallest concave upper-bound of R, it follows that

$$R(q) \leq \overline{R}(q), \quad \forall q \in [0,1].$$

Since $\hat{y}$ is weakly decreasing, the derivative of $\hat{y}$ is weakly negative:

$$\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right) \leq 0, \quad \forall q \in [0,1].$$

Thus,

$$-\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right) \geq 0, \quad \forall q \in [0,1],$$

and

$$-\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right) R(q) \leq -\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right) \overline{R}(q), \quad \forall q \in [0,1].$$

Therefore,

$$\mathop{\mathbb{E}}_{q \sim U(0,1)}\left[-R(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right] \leq \mathop{\mathbb{E}}_{q \sim U(0,1)}\left[-\overline{R}(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right].$$

1.4 *Corollary*

By Parts I and II,

$$\mathop{\mathbb{E}}_{v \sim F}[\overline{\varphi}(v)\hat{x}(v)] = \mathop{\mathbb{E}}_{q \sim U(0,1)}\left[-\overline{R}(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right].$$

and

$$\mathop{\mathbb{E}}_{v \sim F}[\varphi(v)\hat{x}(v)] = \mathop{\mathbb{E}}_{q \sim U(0,1)}\left[-R(q)\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\right].$$

so that

$$\mathop{\mathbb{E}}_{v \sim F}[\overline{\varphi}(v)\hat{x}(v)] - \mathop{\mathbb{E}}_{v \sim F}[\varphi(v)\hat{x}(v)] = \mathop{\mathbb{E}}_{q \sim U[0,1]}\left[-\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\left(\overline{R}(q) - R(q)\right)\right].$$

Now, recall that the revenue curve is "concavified" by correcting non-concavities with line segments that connect their peaks. The corresponding virtual value functions are flat in these regions, because the virtual value function is the derivative of the revenue curve, and the derivative of a line segment is a constant. Correspondingly, the allocation function is also flat in these regions: i.e., its derivative is zero. But if allocation rule $\hat{y}$ is flat wherever ironed revenue exceeds revenue, then $\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right) = 0$, so that

$$\mathop{\mathbb{E}}_{q \sim U[0,1]}\left[-\left(\frac{\mathrm{d}\hat{y}(q)}{\mathrm{d}q}\right)\left(\overline{R}(q) - R(q)\right)\right] = 0.$$

Therefore, expected ironed virtual welfare does not only upper bound expected virtual welfare, these quantities are in fact equal. In other words, constraining the allocation rule to be monotone by ironing does not negatively impact optimal revenue.