The Clinching Auction

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We introduce a new multiparameter setting, for which we can find an approximately welfare-maximizing EPIC auction. We prove the EPIC property by making use of our recipe for doing so: 1. we prove sincere bidding in the auction yields an approximate VCG outcome, and 2. we show consistent bidding strategies dominate inconsistent ones. Not only does this mechanism satisfy desired performance and incentive guarantees (up to some additive error), it is also tractable.

1 *Diminishing Marginal Valuations*

We introduce a new instance of the multiparameter setting, so-called **diminishing marginal valuations** for homogeneous goods. Rather than additive or unit-demand valuations, here each bidder's marginal value for an additional copy of the good is weakly decreasing.

- We assume n bidders and m homogeneous (i.e., identical) goods, with bidders indexed by i, and goods, by j.
- Each bidder i has a marginal value $\mu_i(j)$ for her jth copy of the good, meaning her value for acquiring a jth copy of the good, given she already have $j-1$ copies in its possession.
- Each bidder i's marginal values are weakly decreasing: $\mu_i(1) \geq \mu_i(2) \geq \cdots \geq \mu_i(m)$.

Our goal is to construct an approximately welfare-maximizing EPIC multiunit ascending auction for this scenario.

2 *A Direct Mechanism: A Sanity Check*

Designing a welfare-maximizing DSIC direct mechanism "reduces to" designing a welfare-maximizing EPIC indirect mechanism, in the sense that a polynomial-time solution to the latter can be used to construct a polynomial-time solution to the former via the revelation principle. Therefore, solving for a welfare-maximizing EPIC indirect mechanism is at least as hard as solving for a welfare-maximizing DSIC direct mechanism. As a result, if no such DSIC direct mechanism exists (one that is welfare-maximizing in polynomial time), neither can such an EPIC indirect mechanism.

As a result of this argument, before we embark upon the design of an EPIC indirect mechanism that maximizes welfare in polynomial time, we best do a quick sanity check: can we design a DSIC direct mechanism that maximizes welfare in polynomial time?

Fortunately, we can solve this problem in the affirmative in the direct setting. In particular, the welfare-maximizing allocation can be computed via a simple greedy allocation algorithm:

- Collect a vector of bids $\mathbf{b}$ from all bidders $i \in [n]$, with bid $b_i(j)$ representing i's bid on the jth copy of the good.
- Sort the bids, and then allocate the goods to the bidders who submitted the highest m bids, breaking ties arbitrarily.

For example, if $b_i(4)$ is among the highest m bids, but $b_i(5)$ is not, then bidder i is allocated four copies, which we denote as $x_i = 4$.

As usual, to achieve the VCG outcome, we combine this allocation algorithm with payments that charge bidders their externalities. For each bidder i, we sort the bids by *bidders other than bidder i* from greatest to least, and then establish the following groupings.

$$\underbrace{\beta_1 \quad \beta_2 \quad \cdots \quad \beta_{m-x_i}}_{A} \quad \underbrace{\beta_{m-x_i+1} \quad \beta_{m-x_i+2} \quad \cdots \quad \beta_m}_{B} \quad \underbrace{\beta_{m+1} \quad \beta_{m+2} \quad \cdots \quad \beta_{mn}}_{C}$$

The bids in group A are those that are allocated regardless of i's presence. The bids in group C are those that are *not* allocated regardless of i's presence. The bids in group B are those whose allocation depends on i's presence. These bids comprise bidder i's externality. We therefore charge bidder i, in total, for all x_i copies in group B, the sum of these x_i bids: i.e.,

$$p_i(x_i) = \sum_{j=1}^{x_i} \beta_{m-x_i+j}.$$

By charging each bidder its externality, we charge the VCG payments, thereby guaranteeing the DSIC property.

Example 2.1. Here is an example from the paper that introduced the clinching auction,[1] the subject of this lecture. It is loosely based on the first US Nationwide Narrowband spectrum auction in where there were five bidders and five licenses.

[1] Lawrence M. Ausubel. An efficient ascending-bid auction for multiple objects. *American Economic Review*, 94(5):1452–1475, December 2004

The bidders' marginal values are as follows:

License	A	B	C	D	E
First	123	74	125	84	44
Second	113	5	125	64	24
Third	103	3	49	7	5

If we interpret the values in this table as bids, and sort them from high to low, we arrive at 125, 125, 123, 113, 103, etc., which implies that C wins two licenses, and A wins three. It remains to determine the VCG payments associated with this allocation.

Sorting the bids *sans* C yields: 123, 113, 103, 84, 74, etc. As C's two bids of 125 displace D, who bid 84, and B who bid 74, C's VCG payments are 84 and 74.

Similarly, sorting the bids *sans* A yields: 125, 125, 84, 74, 64, etc. As A's three bids of 123, 113, and 103 displace D, who bid 84, B who bid 74, and D again who bid 64, A's VCG payments are 84, 74, and 64.

Once again, when bidder i is allocated x_i copies of the good, its VCG payment is the sum of x_i bids, one per copy of the good. If we imagine adding i's bids to the sorted list of all other bidders' bids, *one at a time*, each additional winning bid placed by i displaces another lower bid. The smallest bid that i displaces can be viewed as i's payment for its first copy of the good; the second-smallest bid is then i's payment for its second copy of the good; and so on. Payments for additional copies of a good are thus *weakly increasing* (even though values are—by assumption—weakly decreasing).

Building on these observations, we can express bidder i's VCG payment for good j in terms of the other bidders' demands. Define bidder k's **demand set** at price q, $D_k(q) = \max_j \{ j \leq m \mid \mu_k(j) \geq q \}$. Now, bidder i's payment for the jth copy is given by:

$$p_i(j) = \inf \left\{ q \;\middle|\; \sum_{k \neq i} D_k(q) \leq m - j \right\}. \tag{1}$$

Were the payment set at a price any lower than this one, the demand of all bidders other than i would jump to $m - j + 1$, rather than just $m - j$. In other words, this payment is the smallest price at which the demand of all bidders other than i is no more than $m - j$.

As expected, these payments are weakly increasing in j. Each additional copy of the good costs *no less* than the previous, as other bidders' demands fall as the price rises.

3 *An Indirect Mechanism: The Clinching Auction*

Having satisfied the precondition for potential success, we now set our sights on a welfare-maximizing EPIC ascending auction. To this end, we present the **clinching auction**:

- Initialize $q = 0$.
- Collect demand sets from all bidders. (Initially, when $q = 0$, it should be that $D_i(q) = m$, for all bidders i.)
- Alternate between incrementing q by ϵ and collecting demand sets.

- Termination rule: $\sum_{i=1}^{n} D_i(q) \leq m$.
- Activity rule: Ensure that no bidder's demand increases over time: i.e., bidder's demands can only decrease as prices increase.
- Let q^* denote a price at which $\sum_{i\in[n]} D_i(q^*) \leq m - j < \sum_{i\in[n]} D_i(q^* - \epsilon)]$, for some $j \in goods$. (If $\sum_{i\in[n]} D_i(q^* - \epsilon)]$ were equal to m, then the auction would have terminated at price $q^* - \epsilon$.) We can thus allocate y_i copies to all bidders i for which $y_i \in [D_i(q^*), D_i(q^* - \epsilon)]$.

  One way to achieve such an allocation is to allocate to bidder i her demand, namely $D_i(q^*)$ copies. Then, if any unallocated goods remain, they can be allocated at random to any bidders with leftover demand at price $q^* - \epsilon$: i.e., bidders i for whom $D_i(q^* - \epsilon) - D_i(q^*) > 0$.
- Charge bidder i (within ϵ of) its externality. Specifically, charge bidder i for its jth copy of the good:

$$q_i(j) = -\epsilon + \min_{t\in\mathbb{Z}^+} \left\{ \epsilon t \;\middle|\; \sum_{k\neq i} D_k(\epsilon t) \leq m - j \right\}. \tag{2}$$

  As intended, this price is (near) the price at which the demand of all other bidders falls below $m - j$.

  N.B. When $\sum_{k\in[n]} D_k(\epsilon t) = m$, it is not necessary to subtract ϵ from ϵt. But when $\sum_{k\in[n]} D_k(\epsilon t) < m$, the situation is analogous to the last remaining bidders all dropping out at the same time in an English or a Japanese auction, in which case the (one) good is allocated at the final price less ϵ to ensure individual rationality.

Example 3.1. Continuing the setup in Example 2.1 and assuming $\epsilon = 1$, the clinching auction proceeds as follows, with demands depicted only at the most relevant prices:

Price	A	B	C	D	E
10	3	1	3	2	2
25	3	1	3	2	1
45	3	1	3	2	0
50	3	1	2	2	0
65	3	1	2	1	0
75	3	0	2	1	0
85	3	0	2	0	0

At price 65, the aggregate demand of all bidders other than bidder A falls below the total supply of 5. Hence, bidder A "clinches" its first license at this price.[2] The license is "clinched," because the fact

[2] Technically, the price should be 65 less ϵ, but we ignore this adjustment factor.

that the other bidders' demands have fallen below 5 guarantees that bidder A must win this license.

At price 75, the aggregate demand of all bidders other than bidder A falls below 4, so A clinches its second license at this price. In addition, the aggregate demand of all bidders other than bidder C falls below 5, so C clinches its first license at this price.

Note that the auction maintains separate counters for each bidder: i.e., for A, the aggregate demand of others falls below 4, while for C, it falls below 5. Bidder A displaced its first bidder, namely D, at price 65, and is displacing its second bidder, namely B, at price 75, while bidder C is displacing its first bidder, again B, at price 75.

The auction terminates at price 85, when bidder A clinches its third license, and bidder C, its second, both displacing bidder D, and total demand meets total supply. In sum, bidder A pays $65 + 75 + 85$ for its three licenses, and bidder C pays $75 + 85$ for its two licenses. As expected, prices on additional licenses are weakly increasing. (In fact, in this example, they are strictly increasing.)

The outcome of the clinching auction in this example (and always; see Proposition 4.1) is efficient (up to $m\epsilon$). In contrast, in this example, a uniform-price auction[3] would *not* have yielded an efficient outcome, as it would have been in bidder A's best interest to decrease its demand to two licenses when the price reached $75 rather than win all three licenses for $85 each. Winning only two licenses, A's utility would have been $236 - 2(75) = 86$, whereas winning all three, A's utility would have been $339 - 3(85) = 84$. A uniform-price auction is thus susceptible to **strategic demand reduction**.

[3] A uniform-price auction for multiple copies of a homogeneous good is one that charges the same price for all copies of the good.

4 *The Clinching Auction is EPIC*

To prove the clinching auction is approximately EPIC, we follow the design recipe for EPIC auctions. That is, we first show that the outcome of sincere bidding in the clinching auction is approximately VCG (both allocation and payments; Steps 1 and 2 of the design recipe, respectively). We then show that no inconsistent deviations are preferable to sincere bidding (Step 3).

We first show that any allocation the clinching auction attains is welfare maximizing up to $m\epsilon$ (Step 1).

Proposition 4.1. *Assuming sincere bidding, the clinching auction yields total welfare within* $m\epsilon$ *of the optimal.*

En route to showing that the payments in the clinching auction are close to those of VCG (Step 2), we first show that the clinching auction is IR: i.e., no bidder earns negative utility by participating.

Lemma 4.2. *The clinching auction is IR.*

Proof. Assume bidder i wins j copies of the good in the clinching auction, with final price q^*. By the definition of demand sets, $j \leq D_i(q^* - \epsilon)$ iff $\mu_i(j) \geq q^* - \epsilon$. Further, by the design of the payment rule (Equation 2), $q_i(j) \leq q^* - \epsilon$, for all bidders i and all copies of the good j. Therefore, $\mu_i(j) \geq q_i(j)$, for all bidders i and goods j. □

Theorem 4.3. *Given a multiparameter auction setting in which all bidders' have diminishing marginal valuations, the difference in utility earned by a truthful bidder in the VCG auction, assuming others also bid truthfully, and the same bidder bidding sincerely in the clinching auction, assuming others also bid sincerely, is at most $m\epsilon$.*

Having completed Steps 1 and 2 of the EPIC auction design recipe, we have established that sincere bidding in the clinching auction is an EPNE up to $m\epsilon$, among consistent strategies. The remaining piece of this puzzle, then, is to further show that sincere bidding in the clinching auction is an EPNE up to $m\epsilon$, among consistent *and inconsistent* strategies: i.e., that no inconsistent deviations would yield substantially greater utility than sincere bidding. This claim is established in the following theorem.

Theorem 4.4. *The clinching auction is EPIC, up to $m\epsilon$.*

Sketch. Assume all bidders except bidder i bid sincerely. As a result, other bidders' behaviors are not impacted by i's bidding strategy. Moreover, i's payments are determined entirely by the other bidders' demands, which again, i cannot influence. Under these circumstances, we argue that i cannot benefit from bidding inconsistently.

To bid inconsistently in the clinching auction would be to report false demand sets. For example, a bidder might report 1 3 2 3 1 0. But these false reports are dominated by 3 3 3 3 1 0, the downward monotone closure of the false reports, because reporting higher demand earlier can only lead to clinching more units at lower prices. Moreover, 3 3 3 3 1 0 is consistent, for some valuation. So bidding consistently dominates bidding inconsistently.

We have already established that bidding sincerely in the clinching auction is an EPNE up to $m\epsilon$, among consistent strategies. As all inconsistent strategies are dominated by consistent ones, it is more generally EPIC up to $m\epsilon$. □

The argument as to why bidding according to any valuation other than its own cannot benefit i is the usual one.

- If i bids as if its value for a copy of the good were less than it actually is (i.e., it does not demand the copy, even though the price is below its marginal value), and if they don't win the copy as a result (someone else clinches it instead), they are unhappy.

- If *i* bids as if its value for a copy of the good were more than it actually is (i.e., it reports demand for that copy at a price above its marginal value), and if they win (clinch) the copy as a result (at its high price), they are again unhappy.

In sum, inconsistent bidding, meaning bidding in some round as if its valuation were different than it actually is, cannot improve *i*'s utility over consistent bidding.

References

[1] Lawrence M. Ausubel. An efficient ascending-bid auction for multiple objects. *American Economic Review*, 94(5):1452–1475, December 2004.