Competitive Equilibrium and Gross Substitutes

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These lecture notes are closely based on two lectures from Professor Tim Roughgarden's Frontiers in Mechanism Design (CS 364B) course:

- Lecture 5: The Gross Substitutes Condition
- Lecture 6: Gross Substitutes: Welfare Maximization in Polynomial Time

1 *Kelso-Crawford Auction*

The Kelso-Crawford (KC) auction[1] is an ascending auction that generalizes the CK (Crawford Knoer) auction. The auction is formally described in Lecture 5, pp. 2–3. We reproduce the design here, for completeness:

- Initialize the price q_j of all goods j to zero.
- Initialize all bidders' allocations to $\emptyset$: i.e., at the start, no good is allocated to any bidder.
- Repeat forever:
  - Issue demand queries to all bidders. Specifically, ask each bidder to report one[2] of her preferred bundles T_i at prices $\mathbf{q}_\epsilon$: i.e.,

    $$T_i \in \arg\max_{T \subseteq [m] \setminus S_i} \{v_i(S_i \cup T) - \mathbf{q}_\epsilon(S_i \cup T)\}$$

    where

    $$\mathbf{q}_\epsilon(S_i \cup T) = \sum_{j \in S_i} q_j + \sum_{j \in T} (q_j + \epsilon)$$

  - If no bidder reports any demands (i.e., all bidders demand $\emptyset$ at prices $\mathbf{q}_\epsilon$), then terminate the auction with the current allocation and prices. (I.e., make the current tentative allocation permanent.)
  - Otherwise, pick a bidder i for which $T_i \neq \emptyset$, and:
    - (Tentatively) Assign i her preferred bundle: $S_i \leftarrow S_i \cup T_i$.
    - For all bidders $k \neq i$, $S_k \leftarrow S_k \setminus T_i$
    - For all goods $j \in T_i$, $q_j \leftarrow q_j + \epsilon$: i.e., increase the price of good j from q_j to $q_j + \epsilon$ (the price at which i reported j to be an element of one of her preferred bundles).

Example 1.1. Consider an environment with two bidders, 1 and 2, and two goods, A and B. The valuation of the first bidder exhibits complements: $v_1(\{A, B\}) = 3$, while $v_1(\{A\}) = v_1(\{B\}) = 0$. The valuation of the second bidder is unit demand: $v_2(\{A\}) = v_2(\{B\}) = v_2(\{A, B\}) = 2$.

[1] Alexander Kelso and Vincent Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982

[2] Note that ties are broken arbitrarily. It is remarkable that conflicts can still be resolved (i.e., an equilibrium still ensues), even with arbitrary resolution along the equilibrium path.

Consider running the KC auction in this environment. Initially, at price 0, both bidders bid: bidder 1 bids on both goods, while bidder 2 bids on one or the other. One bidder is selected as a tentative winner, say bidder 1, at which point both goods are allocated to bidder 1, and the prices of both goods rise by ϵ. During the next round, only bidder 2 bids; bidder 1 is content with her allocation. Bidder 2 is thus selected as a tentative winner: one of the goods is reallocated from bidder 1 to bidder 2, and the price of that good rises by ϵ. Bidder 1 bids during the next round, and the good that was just allocated to bidder 2 is returned to bidder 1, and as usual, the price of that good rises by ϵ. Continuing as before, only bidder 2 bids in the next round.

This process continues until the price reaches $3/2$.[3] At this point, bidder 1's calculation changes slightly, as he is now facing the prospect of negative utility. His alternatives are to continue bidding so as to win both goods, which yields utility value -2ϵ, or to drop out of the auction, ceding one of the goods to bidder 1, which yields utility value $-3/2$. He chooses to continue bidding, as $-2\epsilon > -3/2$, so the auction continues.

[3] Assume ϵ divides $3/2$ evenly. If it does not, the rest of the discussion of this example would require various ϵ offsets, but the substance would not change.

Pretty soon, the prices of both goods reach 2, at which point bidder 2 drops out. Bidder 1 is thus left with both goods, each at a price of 2 or $2 + \epsilon$, and thus negative utility, since his value for this bundle is only 3.

2 *Gross Substitutes*

Definition 2.1. A combinatorial valuation v_i satisfies the **gross substitutes** condition iff for all prices $\mathbf{p}$, for all bundles $S \in D_i(\mathbf{p})$, and for all prices $\mathbf{q} \geq \mathbf{p}$, there exists $T \subset [m]$ such that $(S \setminus A) \cup T \in D_i(\mathbf{q})$, where $A = \{j \in [m] \mid \mathbf{q}_j > \mathbf{p}_j\}$.

The combinatorial valuation structures we have analyzed thus far—additive, diminishing marginal values, and unit demand—all satisfy the gross substitutes condition.

Proposition 2.2. *Additive valuations satisfy gross substitutes.*

Proposition 2.3. *Unit-demand valuations satisfy gross substitutes.*

Proposition 2.4. *Assuming identical goods, the gross substitutes condition reduces to diminishing marginal values.*

An alternative gross substitute valuation that we have not yet encountered is the k-unit-demand valuation:

$$v_i(S) = \max_{T \subseteq S \mid |T| \leq k} \sum_{j \in T} v_{ij}$$

This valuation generalizes unit-demand: bidders are not restricted to only one their one favorite goods; they are restricted to their favorite bundle of size k.

Note that max is a convex function; moreover, the max of the sum, as in the definition of k-unit valuations, is also convex. Indeed the composition of any convex function with $\sum_{j \in T} v_{ij}$ yields a gross substitutes valuation.

3 Competitive Equilibrium

Theorem 3.1. *If all bidders' valuations satisfy gross substitutes, then the KC auction terminates at an $m\epsilon$-Walrasian equilibrium.*

Proof. The proof relies on three properties of the KC auction, assuming gross substitutes.

- Monotonicity of demand: A price increase on j, with the price of k fixed, can only push demand towards k, not away from k; the demand for k cannot decrease. In particular, bidders never lose interest in goods whose price did not increase.
- Monotonicity of auction prices: As prices increase, the set of bidders interested in a set of goods can only decrease, until this set, and the prices, both stabilize.
- Termination: the auction eventually reaches a state with no overdemand: Prices increase when a good is overdemanded. As the price of a good increases, bidders demand for that good can only decrease. Because demand is monotonic, the number of bidders interested in each good is eventually at most 1, so that there is no longer any overdemand.

These three properties ensure WE0, WE1, and WE2. (**Exercise**.) □

Corollary 3.2. *If all bidders' valuations satisfy gross substitutes, then a Walrasian equilibrium exists.*

Sketch. Run the KC auction repeatedly with $\epsilon = 1/t$ for $t \in \{1, 2, \ldots\}$. Doing so generates a sequence of demands that satisfy $O(\epsilon)$-WE1, as each bidder's intermediate demand is ϵ-optimal, because its demand at the previous increment was optimal and prices could have only increased by ϵ, so the previous demand set can only be suboptimal by up to ϵ. Moreover, the final prices are ϵ-feasible, meaning within ϵ of prices with no overdemand: i.e., overdemand would disappear entirely with an additional ϵ increase in prices. Therefore, KC yields an $O(\epsilon)$-WE.

Now, let $\epsilon \to 0$. Since there are only finitely many allocations, some allocation $S^* = (S_1, \ldots, S_n)$ repeats infinitely often. Moreover, prices are bounded within $[0, \overline{v}]$, so we can extract a subsequence that converges to $\mathbf{p}^*$. Together $(S^*, \mathbf{p}^*)$ comprise a Walrasian equilibrium. □

Example 3.3. Recall Example 1.1. Earlier, we observed that sincere bidding in the KC auction does not lead to a Walrasian equilibrium in this example, as bidder 1's utility is negative, thus violating WE1. Now, we observe that KC could not have possibly landed at a Walrasian equilibrium in this example, because one does not exist.

Recall the first welfare theorem of economics: if an allocation participates in a Walrasian equilibrium, then it is necessarily welfare maximizing. Bidder

1 is thus allocated both goods at any Walrasian equilibrium. For any supporting Walrasian equilibrium prices, the price of this bundle can be at most 3, the value of this bundle to bidder 1. As a result, one of the goods must be priced below 2, but any price below 2 for either good violates WE1 for bidder 2. Therefore, a Walrasian equilibrium does not exist.

4 *EPIC Auctions*

We now return to our regularly scheduled program: the design of EPIC auctions. Recall from the EPIC auction recipe that VCG payments are necessary to satisfy game-theoretic incentive properties.

Example 4.1. Consider an environment with two bidders, 1 and 2, and two goods, A and B. The valuation of the first is additive: $v_1(\{A\}) = v_1(\{B\}) = 2$ and $v_1(\{A, B\}) = 4$. The valuation of the second is unit demand: $v_2(\{A\}) = v_2(\{B\}) = v_2(\{A, B\}) = 1$.

The unique welfare-maximizing allocation in this environment allocates both goods to bidder 1. Bidder 1's VCG payment, i.e., the externality bidder 1 imposes on bidder 2, is 1—for both goods! This VCG outcome does not constitute a Walrasian equilibrium, however.

By the first welfare theorem of economics, if an allocation participates in a Walrasian equilibrium, then it is necessarily welfare maximizing. Bidder 1 is thus allocated also both goods at any Walrasian equilibrium. Walrasian equilibrium prices $\mathbf{p}^*$, however, necessarily exceed bidder 1's VCG payment, as $p_1^*, p_2^* \in [1, 2]$. In particular, $p_j^* \geq 1$ for both goods, which implies that the Walrasian equilibrium price for the bundle $\{A, B\}$ is at least 2, which exceeds bidder 1's VCG payment, which is only 1.

As VCG payments are necessary in an EPIC auction design, this example shows that the KC auction does not yield VCG prices for all gross substitute valuations. More generally, we state the following theorem without proof:

Theorem 4.2. *There is no ascending auction for which sincere bidding yields the VCG outcome for all gross substitute valuations.*

5 *Gross Substitutes as the Limit of Polynomial-Time Computation*

Primal to compute the welfare maximizing allocation:

$$\begin{aligned}
\max_{X \in \mathbb{R}^{n \times 2^m}} \quad & \sum_{i \in [n]} \sum_{S \subseteq [m]} v_{iS} x_{iS} && && (1) \\
\text{subject to} \quad & \sum_{S \subseteq [m]} x_{iS} \leq 1 && \forall i \in [n] && (2) \\
& \sum_{i \in [n]} \sum_{S \subseteq [m] | j \in S} x_{iS} \leq 1 && \forall j \in [m] && (3) \\
& x_{ij} \in \{0, 1\} && \forall i \in [n], j \in [m] && (4)
\end{aligned}$$

Cannot allocate more than one bundle to any one bidder.

Cannot allocate the same good to more than one bidder.

Relax the final set of constraints from integrality constraints to continuous constraints: $x_{ij} \in [0,1] \quad \forall i \in [n], j \in [m]$.

Construct the dual with variables for all constraints in the primal, so "utility" variables for all constraints on the buyers, and "price" variables for all constraints on the goods:

$$\min_{\mathbf{u}\in\mathbb{R}^n, \mathbf{p}\in\mathbb{R}^m} \quad \sum_{i\in[n]} \mathbf{u}_i + \sum_{j\in[m]} \mathbf{p}_j \tag{5}$$

$$\text{subject to} \quad \mathbf{u}_i + \sum_{j\in S} \mathbf{p}_j \geq v_{ij} \qquad \forall i \in [n], \forall S \subseteq [m] \tag{6}$$

$$\mathbf{u}_i, \mathbf{p}_j \geq 0 \qquad \forall i \in [n], j \in [m] \tag{7}$$

Recall strong duality, and the complementary slackness conditions, a consequence of strong duality:

Theorem 5.1. *The primal solution $\boldsymbol{X}^*$ and the dual solution $(\mathbf{u}^*, \boldsymbol{p}^*)$ are both optimal for their respective problems, with equal objective values, iff the complementary slackness conditions hold:*

CS1 $v_{iS} = \mathbf{u}_i + \sum_{j\in S} \mathbf{p}_j$ *whenever* $x_{iS} > 0$

CS2 $\sum_{S\subseteq[m]} x_{iS} = 1$ *whenever* $u_i > 0$

CS3 $\sum_{i\in[n]} \sum_{S\subseteq[m]|j\in S} x_{iS} = 1$ *whenever* $p_j > 0$

Theorem 5.2. *1. There exists a Walrasian equilibrium iff the linear relaxation of WD yields an integral optimal solution. 2. In this case, $\mathbf{p}$ are Walrasian equilibrium prices iff they participate in a solution to the dual, meaning they support some welfare-maximizing $\boldsymbol{X}^*$ that solves the primal.*

Proof. TODO (using Theorem 5.1)! □

Corollary 5.3. *In an environment with only gross substitutes valuations, the linear relaxation is sufficient: i.e., it yields an integral optimal solution.*

Valuations beyond gross substitutes are a thorn in our side . . .

References

[1] Alexander Kelso and Vincent Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982.