# *The CK Auction*

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These lecture notes are closely based on two lectures from Professor Tim Roughgarden's Frontiers in Mechanism Design (CS 364B) course:

- Lecture 2: Unit Demand Bidders and Walrasian Equilibria
- Lecture 3: The Crawford-Knoer Auction

## 1 *Unit-Demand Valuations*

The next two lectures pertain to the **unit-demand** setting, in which bidders are only interested in acquiring one good, but have preferences over which good that might be. The goal of these lectures is to develop an EPIC ascending auction for this setting.

Before attempting to develop an EPIC ascending auction, we first undertake the necessary sanity check—we confirm that there exists a polynomial-time DSIC direct mechanism, assuming unit-demand bidders—because designing the former is at least as hard as designing the latter. Indeed, there exists a polynomial-time algorithm that computes the VCG outcome for the unit-demand setting.

The **winner determination problem**—finding a welfare-maximizing allocation—can be solved in polynomial time assuming unit-demand bidders, by representing bidders and goods as a bipartite graph and solving for the maximum-weight matching. To compute VCG payments, we remove each bidder $i$ from the graph in turn, and re-match the others to goods to find $i$'s externality (Lecture 2, pp. 1–3).

Finding a maximum-weight matching in a bipartite graph, while polynomial, is still non-trivial. The question we are asking in this lecture is whether there exists a simple ascending-price auction that can recover a VCG outcome, and hence do the work of a matching algorithm (actually, $O(n^2)$ runs of a matching algorithm).

## 2 *Crawford-Knoer (CK) Auction*

The Crawford-Knoer (CK) auction[1] is an ascending auction for the unit-demand setting. The auction is formally described in Lecture 3, pp. 2–3. We reproduce the design here, for completeness:

[1]

- Initialize the price $q_j$ of all goods $j$ to zero.
- Initialize all bidders' allocations to $\emptyset$: i.e., at the start, no good is allocated to any bidder.

- Repeat forever:
  - Issue demand queries to all bidders. Specifically, ask each bidder to report one[2] of her preferred goods at prices $\mathbf{q} + \boldsymbol{\epsilon}$: i.e., $j \in D_i(\mathbf{q} + \boldsymbol{\epsilon}) \doteq \arg\max_{j \in G}\{v_i(j) - (q_j + \epsilon)\}$.
  - If no unassigned bidder reports any demands (i.e., all unassigned bidders demand $\emptyset$ at prices $\mathbf{q} + \boldsymbol{\epsilon}$), then terminate the auction with the current allocation and prices.
  - Otherwise, pick an unassigned bidder $i$, and:
    * Assign $i$ her preferred good $j$.
    * Mark whoever was previously assigned $j$ as now unassigned.
    * Increase the price of good $j$ from $q_j$ to $q_j + \epsilon$ (the price at which $i$ reported $j$ to be one of her preferred goods).

[2] Note that ties are broken arbitrarily. It is remarkable that conflicts can still be resolved (i.e., an equilibrium still ensues), even with arbitrary resolution along the equilibrium path.

In this lecture, we follow the EPIC auction design recipe outlined in EPIC Ascending Auctions to prove that CK is an EPIC auction:

1. Design an auction whose allocation rule is welfare maximizing, assuming sincere bidding.
2. Show that sincere bidding also yields VCG payments.
3. Assuming others bid sincerely, argue that no inconsistent bidding strategy is a profitable deviation from likewise bidding sincerely.

## 3 Step 1

Recall that a WE is an allocation and pricing such that each bidder is allocated a preferred good at the current prices (WE1); and a good is priced at 0, if it is unallocated (equivalently, if a good's price is positive, then it is allocated; WE2). We compose two arguments to complete Step 1 of the EPIC design recipe. First, we show that CK, and other similar ascending auctions, such as the KC auction (named for Kelso and Crawford[3]),[4] terminate at a **Walrasian equilibrium** (WE). Second, we invoke (without proof) the first welfare theorem of economics, which states that competitive markets allocate resources efficiently[5]—lending support to Adam Smith's "invisible hand" hypothesis. Together, these two claims establishing that CK (and similar auctions) terminate at a welfare-maximizing allocation.

[3]

[4] See Lecture 5: Gross Substitutes I.

[5] Technically Pareto-efficiently, which is weaker than welfare maximization, but more broadly applicable, beyond the unit-demand setting.

**Proposition 3.1** (Lecture 3, Lemma 3.2)**.** *Assuming sincere bidding, the CK auction terminates at an $m\epsilon$-WE.*

*Proof.* This result is straightforward, given the rules of the auction and the assumption that bidders' valuations are unit-demand.

First, we show WE1: If a bidder $i$ wins a good $j$, it is because $i$ bid on $j$, which can only happen when $j$ is in $i$'s demand set. In other words, at some point in the auction $i$'s utility for $j$ was at least as great as its utility for any other good. Moreover, $j$'s price did not change since the time at which it was tentatively allocated to $i$. But the prices of the other goods may have increased. So $i$'s utility for $j$ as compared to $i$'s utility for the other goods can only have increased. Therefore, $j$ remains in $i$'s demand set, so that WE1 holds for bidder $i$, and holds similarly for all other bidders.

Second, we show WE2: In the CK auction, a bidder cannot bid on a good, cause its price to increase, and then relinquish the good, leaving the price non-zero and the good unallocated. Thus, if a good is not allocated, it can only be because no one ever bid on it, in which case its price is necessarily zero. Thus, WE2 also holds. $\square$

Together with the first welfare theorem of economics,[6] this result implies the following:

[6] A special case of this (far-reaching) theorem, sufficient for our purposes, appears in Lecture 2; Proposition 3.4.

**Corollary 3.2.** *Assuming sincere bidding, the CK auction terminates at an $m\epsilon$-welfare-maximizing outcome.*

## 4 *Step 2*

Recall that VCG payments are an essential component of our EPIC auction design recipe. But fear not: as long as we can establish a relationship between VCG payments and WE prices, we can still establish the requisite incentive guarantees.

In the unit-demand setting, it turns out that every VCG outcome is a WE. Indeed, each one is a smallest WE (i.e., component-wise no greater than any other WE), making VCG outcomes a natural contender for the outcome of an auction with ascending prices.

**Theorem 4.1.** *Assuming bidders with unit demand valuations, VCG payments correspond to prices at a smallest WE.*

This claim follows from two others.

**Proposition 4.2** (Lecture 2, Theorem 3.6)**.** *Assuming bidders with unit demand valuations, VCG payments constitute WE prices.*

**Proposition 4.3** (Lecture 2, Theorem 3.5)**.** *Assuming bidders with unit demand valuations, VCG payments lower bound WE prices.*

Before we prove Proposition 4.2, recall that VCG payments, which are ordinarily associated with bidders, can just as well be associated with goods in a unit-demand setting, since each bidder is allocated at most one good: i.e., VCG payments are effectively anonymous (VCG) prices in unit-demand environments.

*Proposition 4.2.* First, observe that WE2 is satisfied immediately, as VCG prices for unallocated goods are zero.

It remains to show WE1. To do so, we invoke Lemma 3.7, which states that the VCG payment associated with an arbitrary good $j$ in the unit-demand setting can be understood as the welfare gain achieved by injecting an additional copy of $j$ into the market. If good $j$ was allocated to bidder $i$, then this gain is exactly $i$'s VCG payment (i.e., $j$'s price), because injecting an additional copy of good $j$ into the market is akin to removing $i$ from the market.

Now, to show WE1, we will indeed inject a second copy $l'$ of some good $l$ is into the market. Assume that bidder $i$ who was allocated good $j$ before $l'$ was injected into the market would have preferred $l'$, so is now allocated $l'$. Next, we reoptimize for all bidders other than $i$ and for all goods $j$ (although not $l'$, which has been allocated to $i$). In this new extended market, the increase in welfare is at least bidder $i$'s value for $l$, less bidder $i$'s value for $j$, plus the other bidders' gains now that $j$ has become available to them. These gains, however, are precisely bidder $i$'s VCG payment $p_j$, which represents bidder $i$'s externality in the original market. Finally, by Lemma 3.7, the VCG price $p_l$ of good $l$ is the welfare gain achieved by injecting an additional copy of $l$ into the market. But, by the argument above, this welfare gain (i.e., $p_l$) is at least $v_i(l) - v_i(j) + p_j$.[7] In other words, $v_i(j) - p_j \geq v_i(l) - p_l$, for all goods $l$. Since bidder $i$ was allocated good $j$, WE1 is satisfied. □

[7] We have only a lower bound on welfare gain because it may not have been welfare-maximizing to allocate the extra copy of $l$ to $i$.

*Proposition 4.3.* Assume a WE $(M^*, \boldsymbol{q})$ with an allocation given by $M^*$[8] and prices given by $\boldsymbol{q}$. We write $M(i)$ to denote the good allocated to $i$, say $j$, and $q_{M(i)} = q_j$ to denote the $j$'s price.

[8] $M$ stands for matching

First, letting $Q = \sum_{l \in G} q_l$, we observe the following:

1. $\sum_{i \in [n]} q_{M^*(i)} = Q$, since all goods unallocated by $M^*$ are priced at 0, by WE2.

2. $Q' = \sum_{i \in [n]} q_{M'(i)} \leq \sum_{i \in [n]} q_{M^*(i)} = Q$, for some arbitrary alternative allocation $M'$, because $Q'$ can only be less than $Q$ on account of goods allocated in $M^*$ but not in $M'$, while the price of any goods allocated in $M'$ but not in $M^*$ is again 0, by WE2.

Now, let $M$ denote a VCG allocation and let and $p_{M(i)} = p_j$ denote the VCG price associated with $j$. Since $M$ is welfare maximizing, it follows that $(M, \boldsymbol{q})$ is a WE,[9] as all competitive equilibrium prices, such as $\boldsymbol{q}$, "support" all welfare-maximizing allocations, such as $M$.

[9] Homework 7, Problem 2

VCG prices are zero for goods unmatched by $M$, so it suffices to consider matched goods only. Assume bidder $i$ is allocated good $j$ in $M$, and let $M^{-i}$ denote a welfare-maximizing allocation without $i$.

By WE1, for all $k \neq i$, it holds that

$$v_{k,M(k)} - q_{M(k)} \geq v_{k,M^{-i}(k)} - q_{M^{-i}(k)}$$

Summing over all $k \neq i$ yields

$$\sum_{k \neq i} v_{k,M(k)} - \underbrace{\sum_{k \neq i} q_{M(k)}}_{=Q-q_j} \geq \sum_{k \neq i} v_{k,M^{-i}(k)} - \underbrace{\sum_{k \neq i} q_{M^{-i}(k)}}_{\leq Q}$$

Therefore,

$$\sum_{k \neq i} v_{k,M(k)} - (Q - q_j) \geq \sum_{k \neq i} v_{k,M^{-i}(k)} - Q$$

Canceling out the $Q$'s and rearranging yields:

$$q_j \geq \sum_{k \neq i} v_{k,M^{-i}(k)} - \sum_{k \neq i} v_{k,M(k)}$$

But the right-hand side of this equation is precisely $i$'s VCG payment $p_j$. Therefore, for all goods $j \in G$, $q_j \geq p_j$. $\square$

In Step 1 we showed, assuming sincere bidding (and bidders with unit-demand valuations), the CK auction terminates at WE prices up to $m\epsilon$. But then, by Theorem 4.1, it likewise terminates at VCG prices up to $m\epsilon$, because they comprise a smallest WE. Therefore, sincere bidding is an EPNE up to $m\epsilon$ among consistent strategies.

## 5 *Step 3*

The final step in the proof that the CK auction is EPIC is to show that no inconsistent strategy is a profitable deviation from sincere bidding up to some additive error: i.e., no inconsistent strategy yields much greater utility than sincere bidding (Lecture 3, Theorem 4.2).

The proof of this claim depends on two symmetric bounds (Lecture 3, Lemmas 3.4 and 3.5). The first states that the final price of a good in the CK auction, assuming sincere bidding, is upper bounded by its VCG price plus $\delta = \epsilon \min\{m, n\}$, while the second states that the price of a good in the CK auction is lower bounded by its VCG price minus $\delta$: i.e., the final price $q_j$ of good $j$ falls in the range $[p_j - \delta, p_j + \delta]$, where $p_j$ is the VCG price for good $j$. As always, VCG prices are the outcome of truthful bidding in a VCG auction.

The proof now proceeds by showing that any deviation in the CK auction via an inconsistent strategy can be replicated by a sincere one up to $2\delta$. Then, since sincere bidding is an EPNE up to $m\epsilon$ among (only) consistent strategies, sincere bidding is an EPNE up to $\max\{2\delta, m\epsilon\}$ among both consistent and inconsistent strategies, or up to $2m\epsilon$, assuming $m < n$, which is the common case.

*Remark* 5.1. Lemma 3.5 actually assumes something weaker than sincere bidding by all in the CK auction. Rather, it bounds $\epsilon$-WE prices directly (however they might arise).

**Theorem 5.2.** *The CK auction is EPIC up to* $2\delta$.

*Proof.* Assume valuations $(v_i, \mathbf{v}_{-i})$, and that all bidders other than $i$ bid sincerely. We must show that it does not benefit bidder $i$ (very much) to bid inconsistently.

Assume that a CK auction has transpired, with bidder $i$ bidding insincerely. If bidder $i$ does not win anything, then there was no benefit to bidding inconsistently. Thus, it suffices to consider the case where bidder $i$ wins good $j$.

Now consider a CK auction with all bidders, including bidder $i$, bidding sincerely, but assume bidder $i$ has the following valuation, $v'_i$:

$$v'_{ik} = \begin{cases} \infty & k = j \\ 0 & \text{otherwise} \end{cases}.$$

We claim that the outcome of the *original* CK auction, with bidder $i$ bidding inconsistently, is an $m\epsilon$-WE of the market defined by unit-demand bidders with valuations $(v'_i, \mathbf{v}_{-i})$. Why? Well, $i$ won $j$, which is $i$'s favorite good when $i$'s valuation is $v'_i$, and all the other bidders won goods in their demand sets as well, since they were bidding sincerely (WE1), while unallocated goods were priced at zero (WE2).

We complete the argument as follows:

$i$'s utility in the CK auction assuming $i$ bids inconsistently while the others bid sincerely
$= i$'s utility at the $m\epsilon$-WE of the market defined by valuations $(v'_i, \mathbf{v}_{-i})$
$\leq i$'s utility in a VCG auction assuming valuations $(v'_i, \mathbf{v}_{-i})$, plus $\delta$
$\leq i$'s utility in a VCG auction assuming valuations $(v_i, \mathbf{v}_{-i})$, plus $\delta$
$\leq i$'s utility at the $m\epsilon$-WE that arises assuming sincere bidding by all, plus $2\delta$

The first equality follows by the aforementioned claim.

The first inequality follows because bidder $i$'s allocation when she bids inconsistently is the same as it is in a VCG auction assuming valuation $v'_i$. Thus, it suffices to compares prices only, when comparing $i$'s utility in these two settings. But any prices that arise at an $m\epsilon$-WE are at least VCG prices less $\delta$ (by Lemma 3.5, the lower bound, taking into account Remark 5.1).

The second inequality follows from the fact that VCG is DSIC, so it cannot benefit bidder $i$ to bid according to $v'_i$ rather than $v_i$ in a VCG auction, when $i$'s valuation is $v_i$.

The third inequality follows because prices that arise as the outcome of sincere bidding in the CK auction are at most VCG prices plus $\delta$ (by Lemma 3.4, the upper bound on CK auction prices). $\square$