Myerson's Payment Characterization

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We describe Myerson's lemma,[1] in which he characterizes the payment rule that incentivizes truth telling in single-parameter auctions.

[1] Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981

1 *Payment Characterization*

In single-parameter auction, each bidder i's valuation is described by a single parameter v_i, which represents, for example, the bidder's value for a single good. In such an auction, quasi-linear utilities are given by: $u_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i})$.

This lecture concerns a single-parameter auction in which bidder i's values $v_i \in T_i = [\underline{v}_i, \overline{v}_i]$, with lowest and highest types $\underline{v}_i, \overline{v}_i \in \mathbb{R}_+$.

Theorem 1.1 (Myerson)**.** *A single-parameter auction is dominant-strategy incentive compatibile (DSIC) iff the following two conditions hold:*

1. *The allocation rule is monotone in values:*

$$x_i(v_i, \mathbf{v}_{-i}) \geq x_i(t_i, \mathbf{v}_{-i}), \quad \forall i \in [n], \forall v_i \geq t_i \in T_i, \forall \mathbf{v}_{-i} \in T_{-i}. \quad (1)$$

2. *Payments are computed as follows:*

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \left(\int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z + \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i}) \right), \forall i \in [n], \forall v_i \in T_i, \forall \mathbf{v}_{-i} \in T_{-i}. \quad (2)$$

Further, if, for all bidders $i \in [n]$, the utility $u_i(\underline{v}_i, \mathbf{v}_{-i}) \geq 0$, then these two conditions imply that the auction is individually rational (IR) as well.

Myerson's payment formula, Equation (2), is easy to interpret by visualizing it. We begin by drawing a box $v_i x_i(v_i, \mathbf{v}_{-i})$, as in Figure 1. Next, we subtract the area under the allocation curve, namely $\int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z$. We also subtract the box $\underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i})$. These areas are depicted in Figure 2. The remaining area is the payment bidder i makes, assuming $p_i(\underline{v}_i, \mathbf{v}_{-i}) = 0$, as shown in Figure 3. In sum, the payment at a point v_i is simply the area to the left of the allocation function from $x_i(\underline{v}_i, \mathbf{v}_{-i})$ to $x_i(v_i, \mathbf{v}_{-i})$.

Proof. We first prove the if direction, namely that DSIC implies that the allocation rule is monotone and that payments take the form of Equation (2). We start with the first condition (monotonicity).

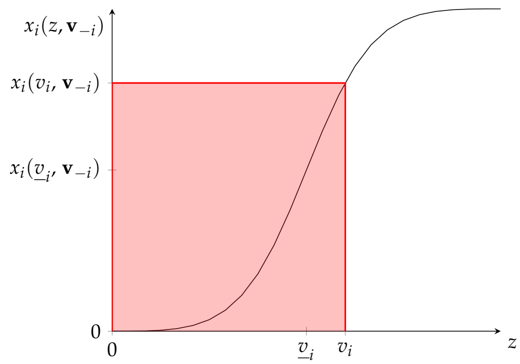

Figure 1: Area $v_i x_i(v_i, \mathbf{v}_{-i})$.

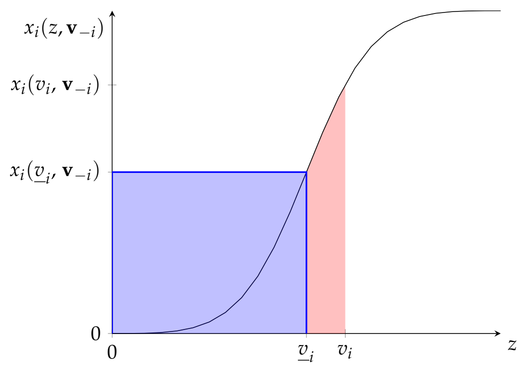

Figure 2: Area $\int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i})\,\mathrm{d}z$ and area $\underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i})$.

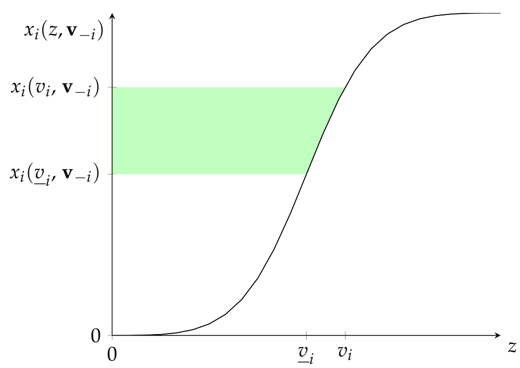

Figure 3: Payment: $v_i x_i(v_i, \mathbf{v}_{-i}) - \left(\int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i})\,\mathrm{d}z + \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_i)\right)$, with $p_i(\underline{v}_i, \mathbf{v}_i) = 0$.

By DSIC, $\forall i \in [n]$, if i's type is $v_i \in T_i$, then for all $t_i \in T_i, \mathbf{v}_{-i} \in T_{-i}$,

$$u_i(v_i, \mathbf{v}_{-i}) \geq u_i(t_i, \mathbf{v}_{-i}).$$

Likewise, if i's type is $t_i \in T_i$, then for all $v_i \in T_i, \mathbf{v}_{-i} \in T_{-i}$,

$$u_i(t_i, \mathbf{v}_{-i}) \geq u_i(v_i, \mathbf{v}_{-i}).$$

Equivalently,

$$\begin{aligned} v_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i}) &\geq v_i x_i(t_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) \\ t_i x_i(t_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) &\geq t_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i}). \end{aligned}$$

Rearrange the expressions to collect payments:

$$\begin{aligned} v_i x_i(v_i, \mathbf{v}_{-i}) - v_i x_i(t_i, \mathbf{v}_{-i}) &\geq p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) \\ p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) &\geq t_i x_i(v_i, \mathbf{v}_{-i}) - t_i x_i(t_i, \mathbf{v}_{-i}). \end{aligned}$$

Combine the expressions to form one inequality:

$$v_i x_i(v_i, \mathbf{v}_{-i}) - v_i x_i(t_i, \mathbf{v}_{-i}) \geq p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) \geq t_i x_i(v_i, \mathbf{v}_{-i}) - t_i x_i(t_i, \mathbf{v}_{-i}).$$

Collect like terms:

$$v_i \left(x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i}) \right) \geq p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i}) \geq t_i \left(x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i}) \right). \tag{3}$$

And drop the payment terms:

$$v_i \left(x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i}) \right) \geq t_i \left(x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i}) \right). \tag{4}$$

If $v_i \geq t_i$, then in order for this inequality to hold, $x_i(v_i, \mathbf{v}_{-i})$ cannot be less than $x_i(t_i, \mathbf{v}_{-i})$. So, the allocation rule must be monotone (1).

Next, we show that payments must take the form of Equation (2). Continuing from Equation 3, we divide each expression by $v_i - t_i$:

$$v_i \left(\frac{x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i})}{v_i - t_i} \right) \geq \left(\frac{p_i(v_i, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i})}{v_i - t_i} \right) \geq t_i \left(\frac{x_i(v_i, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i})}{v_i - t_i} \right).$$

If $v_i \geq t_i$, then we can write v_i as $v_i = t_i + \delta$, for some $\delta \geq 0$:

$$(t_i + \delta) \left(\frac{x_i(t_i + \delta, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i})}{t_i + \delta - t_i} \right) \geq \frac{p_i(t_i + \delta, \mathbf{v}_{-i}) - p_i(t_i, \mathbf{v}_{-i})}{t_i + \delta - t_i} \geq t_i \left(\frac{x_i(t_i + \delta, \mathbf{v}_{-i}) - x_i(t_i, \mathbf{v}_{-i})}{t_i + \delta - t_i} \right). \tag{5}$$

Now that we have functions of form

$$\frac{f(x + \delta) - f(x)}{\delta},$$

we can compute the limit of these functions as $\delta \to 0$ by computing the corresponding derivatives.

By Lebesgue's theorem, the limits of the RHS and the LHS of Equation (5) as $\delta \to 0$ must exist almost everywhere (a.e.).[2] Moveover, these limits are equal a.e.. So by the squeeze theorem, since the upper and lower bounds of the middle expression are equal a.e., the latter must also equal this value a.e.. Finally, we observe that the middle expression corresponds to the derivative of the pricing rule, while the LHS and the RHS of Equation (5) correspond to a function that depends on the derivative of the allocation function. Therefore,

$$z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) = \frac{\mathrm{d}p_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \quad \text{a.e..}$$

Next, we integrate both sides from the lowest to the highest type:[3]

[2] A function is almost everywhere differentiable if its derivative exists everywhere except on a set of measure 0. By Lebesgue's theorem on the differentiability of monotonic functions, the allocation rule x_i is almost everywhere differentiable since its domain, i.e., the type space, is bounded. This weaker notion of differentiability allows us to conclude that both limits exist as $\delta \to 0$, and that they are (Lebesgue) integrable.

[3] By the definition of the Lebesgue integral, the value of the integral of a function on sets of measure 0 is 0. Therefore, integrating both sides of this equality, which was defined only almost everywhere, turns it into an equality that is defined everywhere!

$$\int_{\underline{v}_i}^{v_i} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) \mathrm{d}z = \int_{\underline{v}_i}^{v_i} \frac{\mathrm{d}p_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \mathrm{d}z$$
$$= p_i(v_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i}).$$

We then integrate the left-hand side by parts:

$$\int_a^b u \, \mathrm{d}v = uv|_a^b - \int_a^b v \, \mathrm{d}u,$$

where we let

$$u = z \qquad \mathrm{d}u = \mathrm{d}z$$
$$\mathrm{d}v = \frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z}\mathrm{d}z \qquad v = x_i(z, \mathbf{v}_{-i}),$$

to get

$$\int_{\underline{v}_i}^{v_i} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z} \right) \mathrm{d}z = zx_i(z, \mathbf{v}_{-i})|_{\underline{v}_i}^{v_i} - \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z$$
$$= v_i x_i(v_i, \mathbf{v}_{-i}) - \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z.$$

Therefore,

$$p_i(v_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z,$$

which implies

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \left(\int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z + \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i}) \right).$$

We now prove the only if direction, namely that if the allocation rule is monotone and payments take the form of Equation (2), then the DSIC constraints must hold: i.e., bidding neither $v_i + \delta$ nor $v_i - \delta$, for some $\delta > 0$, is preferable to bidding v_i.

By bidding $v_i + \delta$ for some $\delta > 0$, bidder i's utility,[4] is

[4] up to a constant, namely $p_i(\underline{v}_i, \mathbf{v}_{-i}) - \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i})$.

$$u_i(v_i + \delta, \mathbf{v}_{-i}; v_i) = v_i x_i(v_i + \delta, \mathbf{v}_{-i}) - p_i(v_i + \delta, \mathbf{v}_{-i})$$
$$= v_i x_i(v_i + \delta, \mathbf{v}_{-i}) - \left((v_i + \delta) \, x_i(v_i + \delta, \mathbf{v}_{-i}) - \int_{\underline{v}_i}^{v_i + \delta} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z \right)$$
$$= -\delta x_i(v_i + \delta, \mathbf{v}_{-i}) + \int_{\underline{v}_i}^{v_i + \delta} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z.$$

Comparing utilities between a truthful bid and any higher bid:[5]

[5] The constants cancel.

$$\int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z - \left[-\delta x_i(v_i + \delta, \mathbf{v}_{-i}) + \int_{\underline{v}_i}^{v_i + \delta} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z \right]$$
$$= \delta x_i(v_i + \delta, \mathbf{v}_{-i}) - \int_{v_i}^{v_i + \delta} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z$$

$$\geq 0.$$

This final inequality follows from the monotonicty of the allocation function. For all $\gamma \in [v_i, v_i + \delta]$, $x_i(\gamma, \mathbf{v}_{-i}) \leq x_i(v_i + \delta, \mathbf{v}_{-i})$. Therefore, the integral is upper-bounded by $\delta x_i(v_i + \delta, \mathbf{v}_{-i})$. See Figure 4.

The situation is analogous for $v_i - \delta$. By bidding this amount, bidder i's utility,[6] is

[6] up to the same constant.

$$\begin{aligned} u_i(v_i - \delta, \mathbf{v}_{-i}; v_i) &= v_i x_i(v_i - \delta, \mathbf{v}_{-i}) - p_i(v_i - \delta, \mathbf{v}_{-i}) \\ &= v_i x_i(v_i, \mathbf{v}_{-i}) - \left((v_i - \delta)\, x_i(v_i + \delta, \mathbf{v}_{-i}) - \int_{\underline{v}_i}^{v_i - \delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z \right) \\ &= \delta x_i(v_i - \delta, \mathbf{v}_{-i}) + \int_{\underline{v}_i}^{v_i - \delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z. \end{aligned}$$

Comparing utilities between a truthful bid and any lower bid:[7]

[7] Again, the constants cancel.

$$\begin{aligned} &\int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z - \left[\delta x_i(v_i - \delta, \mathbf{v}_{-i}) + \int_{\underline{v}_i}^{v_i - \delta} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z \right] \\ &= \int_{v_i - \delta}^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z - \delta x_i(v_i - \delta, \mathbf{v}_{-i}) \\ &\geq 0. \end{aligned}$$

This final inequality follows from the monotonicty of the allocation function. For all $\gamma \in [v_i - \delta, v_i]$, $x_i(\gamma, \mathbf{v}_{-i}) \geq x_i(v_i - \delta, \mathbf{v}_{-i})$. Therefore, the integral is lower-bounded by $\delta x_i(v_i - \delta, \mathbf{v}_{-i})$. See Figure 4.

Since $\delta = 0$ is optimal, the DSIC constraints hold.

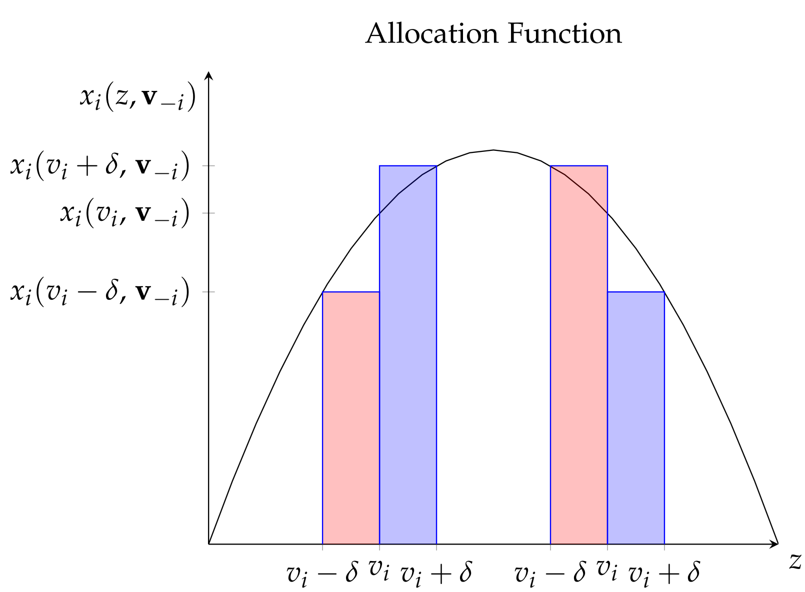

Figure 4: Bidding truthfully vs. not. Bidding truthfully is undominated where the allocation function increases, and is dominated otherwise.

Finally, we show IR. The utility of each bidder $i \in [n]$, according to the payment rule, Equation (2), is

$$\begin{aligned} u_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i}) \\ &= v_i x_i(v_i, \mathbf{v}_{-i}) - \left(v_i x_i(v_i, \mathbf{v}_{-i}) - \left(\int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z + \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i}) \right) \right) \\ &= \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z + \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i}). \end{aligned}$$

To ensure that this final quantity is non-negative requires only that $u_i(\underline{v}_i, \mathbf{v}_{-i}) \geq 0$, for all bidders $i \in [n]$.

More specifically, satisfying this inequality requires that $\underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) \geq p_i(\underline{v}_i, \mathbf{v}_{-i})$, which is achieved, for example, in auctions in which the lowest types $\underline{v}_i \in T_i$ are allocated nothing and pay nothing (i.e., when $x_i(\underline{v}_i, \mathbf{v}_{-i}) = 0$ and $p_i(\underline{v}_i, \mathbf{v}_{-i}) = 0$, for all $\mathbf{v}_{-i} \in T_{-i}$). Notably, as per the payment formula, only winners make payments in all such auctions (when allocation function is 0, so is its integral). $\square$

References

[1] Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.