The Envelope Theorem

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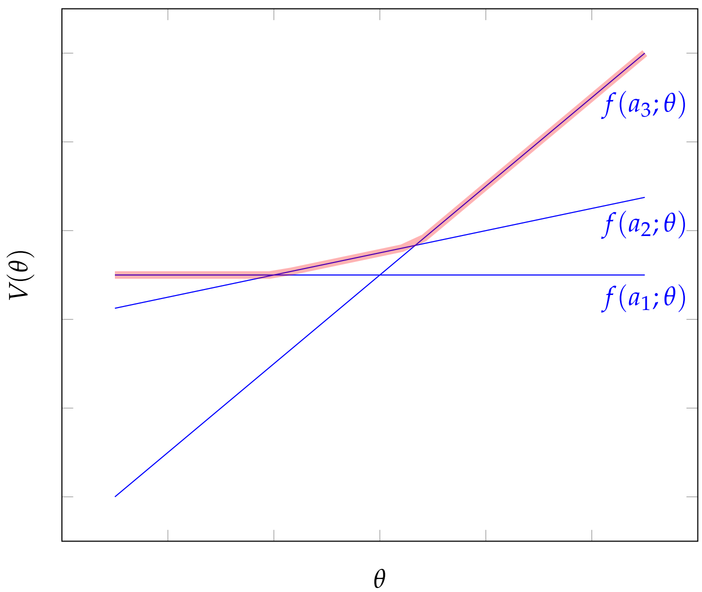
We prove the celebrated envelope theorem. Then, by way of this theorem, we derive the symmetric equilibrium in first-price auctions and Myerson's payment characterization for DSIC auctions. When orinignally drafted, these notes followed the presentation in Quint;[1] by now, there are likely deviations.

[1] Dan Quint. Some beautiful theorems with beautiful proofs. University of Wisconsin–Madison, 2014

1 *Envelope Theorem*

We prove the envelope theorem in its simplest form. In more interesting/ complicated versions, the constraint set depends on the parameter θ.



Consider an optimization problem

$$V(\theta) = \max_{a \in A} f(a;\theta) \ .$$

We write $f(\cdot;\theta)$ to indicate that f is "parameterized" by some $\theta \in \Theta$.[2] This parameterization is intended to indicate that optimization over the set A is in fact an optimization over a strategy space of functions $s : \Theta \to A$ from the parameter space Θ to A.

[2] Such parameterizations are also sometimes denoted with subscripts instead of semicolons: i.e., $f_\theta(\cdot)$.

Theorem 1.1. *Let $A^*(\theta) = \arg\max_{a \in A} f(a;\theta)$, and assume $A^*(\theta)$ is nonempty, with $s^*(\theta)$ an element of $A^*(\theta)$, so that $V(\theta) = f(s^*(\theta);\theta)$. If $V(\theta)$ and $f(a;\theta)$, for all $a \in A$, are differentiable at $\theta \in \Theta$, then*

$$V'(\theta) = \frac{\mathrm{d} \max_{a \in A} f(a;\theta)}{\mathrm{d}\theta} = \frac{\mathrm{d} f(s^*(\theta);\theta)}{\mathrm{d}\theta} \ .$$

Proof. By definition, if V is differentiable at θ, then

$$V'(\theta) = \lim_{\epsilon \to 0} \frac{V(\theta+\epsilon) - V(\theta)}{\epsilon} = \lim_{\epsilon \to 0} \frac{V(\theta) - V(\theta-\epsilon)}{\epsilon} \; .$$

Since $V(\theta+\epsilon) = \max_{a \in A} f(a;\theta+\epsilon) \geq f(s^*(\theta);\theta+\epsilon)$, it follows that

$$V'(\theta) = \lim_{\epsilon \to 0} \frac{V(\theta+\epsilon) - V(\theta)}{\epsilon} \geq \lim_{\epsilon \to 0} \frac{f(s^*(\theta);\theta+\epsilon) - f(s^*(\theta);\theta)}{\epsilon} = \frac{\mathrm{d}f(s^*(\theta);\theta)}{\mathrm{d}\theta} \; .$$

Similarly, since $V(\theta-\epsilon) = \max_{a \in A} f(a;\theta-\epsilon) \geq f(s^*(\theta);\theta-\epsilon)$, it follows that

$$V'(\theta) = \lim_{\epsilon \to 0} \frac{V(\theta) - V(\theta-\epsilon)}{\epsilon} \leq \lim_{\epsilon \to 0} \frac{f(s^*(\theta);\theta) - f(s^*(\theta);\theta-\epsilon)}{\epsilon} = \frac{\mathrm{d}f(s^*(\theta);\theta)}{\mathrm{d}\theta} \; .$$

The result now follows, as

$$\frac{\mathrm{d}f(s^*(\theta);\theta)}{\mathrm{d}\theta} \leq V'(\theta) \leq \frac{\mathrm{d}f(s^*(\theta);\theta)}{\mathrm{d}\theta} \; .$$

□

Example 1.2. The following functions are shown in the figure above:

$$\begin{aligned} f(a_1;\theta) &= 1 \\ f(a_2;\theta) &= \theta/4 + 3/2 \\ f(a_3;\theta) &= \theta + 1 \end{aligned}$$

Observe that $f(a;\theta)$ is differentiable, for all $a \in A$, and at all $\theta \in \Theta$, with

$$\begin{aligned} \frac{\mathrm{d}f(a_1(\theta);\theta)}{\mathrm{d}\theta} &= 0 \\ \frac{\mathrm{d}f(a_2(\theta);\theta)}{\mathrm{d}\theta} &= 1/4 \\ \frac{\mathrm{d}f(a_3(\theta);\theta)}{\mathrm{d}\theta} &= 1 \end{aligned}$$

Although $V(\theta)$ is *not* differentiable at $\{-2, 2/3\}$, the envelope theorem gives:

$$V'(\theta) = \begin{cases} \frac{\mathrm{d}f(a_1(\theta);\theta)}{\mathrm{d}\theta} & \theta < -2 \\ \frac{\mathrm{d}f(a_2(\theta);\theta)}{\mathrm{d}\theta} & -2 < \theta < 2/3 \\ \frac{\mathrm{d}f(a_3(\theta);\theta)}{\mathrm{d}\theta} & 2/3 < \theta \end{cases}$$

Therefore,

$$\begin{cases} 0 & \theta < -2 \\ 1/4 & -2 < \theta < 2/3 \\ 1 & 2/3 < \theta \end{cases}$$

2 *Key Observation*

Recall that an auction is defined by two rules, an allocation rule and a payment rule. We consider a single-parameter auction for one good in which each bidder's i's valuation/value for the good is described by one number, $v_i \in T_i$, based on which she chooses her bid $b_i \in B_i$.

Fixing all other agents' strategies $\mathbf{s}_{-i} : T_{-i} \to B_{-i}$, we abbreviate bidder i's payment when she bids b_i by $p_i(b_i) \doteq p_i(b_i, \mathbf{s}_{-i}(\cdot))$. Her quasilinear utility if she is allocated the good is then $v_i - p_i(b_i)$. Furthermore, since her allocation probability is $x_i(b_i)$, her expected utility is $v_i x_i(b_i) - p_i(b_i)$.

In this setting, the envelope theorem yields an interesting insight about the optimal expected utility function, namely that its derivative is the (expected) allocation function.

Theorem 2.1. *Given a bidder i with value v_i and quasilinear utility function $u_i(b_i) = v_i x_i(b_i) - p_i(b_i)$, so that her* optimal *utility function is given by*

$$U^*(v_i) = \max_{b_i \in B_i} v_i x_i(b_i) - p_i(b_i) \ .$$

The derivative of her optimal utility function with respect to her value is her allocation function: i.e.,

$$\frac{\mathrm{d}U^*(v_i)}{\mathrm{d}v_i} = x_i(b_i^*) \ ,$$

where $b_i^* = s_i^*(v_i)$ denotes a utility-maximizing bid.

Proof. First, letting $f(b_i; v_i) = v_i x_i(b_i) - p_i(b_i)$,

$$U_i'(v_i) = \frac{\mathrm{d}}{\mathrm{d}v_i} \max_{b_i \in B_i} f(b_i; v_i) \tag{1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}v_i} \max_{b_i \in B_i} v_i x_i(b_i) - p_i(b_i) \tag{2}$$

$$= \frac{\mathrm{d}}{\mathrm{d}v_i} v_i x_i(b_i^*) - p_i(b_i^*) \tag{3}$$

Equation 2 follows from the definition of optimal expected utility, while Equation 3 follows via the envelope theorem.

Next, by the sum rule of calculus:

$$\frac{\mathrm{d}}{\mathrm{d}v_i} v_i x_i(b_i^*) - p_i(b_i^*) = \frac{\mathrm{d}}{\mathrm{d}v_i} v_i x_i(b_i^*) - \frac{\mathrm{d}}{\mathrm{d}v_i} p_i(b_i^*) \ . \tag{4}$$

Applying the product rule to the first term of Equation 4 yields:

$$\frac{\mathrm{d}}{\mathrm{d}v_i} v_i x_i(b_i^*) = x_i(b_i^*) + v_i \frac{\mathrm{d}x_i(b_i^*)}{\mathrm{d}v_i} \ . \tag{5}$$

Further simplification now requires the chain rule.

Recall the chain rule: if $y = f(u)$ and $u = g(v)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \frac{\mathrm{d}y}{\mathrm{d}u} \frac{\mathrm{d}u}{\mathrm{d}v} \ .$$

Letting $f = x_i(u)$ and $g = s_i^*(v_i)$, so that $f = x_i(s_i^*(v_i))$, yields:

$$\frac{\mathrm{d}x_i(b_i^*)}{\mathrm{d}v_i} = \frac{\mathrm{d}x_i(b_i^*)}{\mathrm{d}b_i}\frac{\mathrm{d}b_i}{\mathrm{d}v_i} \tag{6}$$

$$= \frac{\mathrm{d}x_i(b_i)}{\mathrm{d}b_i}\frac{\mathrm{d}b_i}{\mathrm{d}v_i}\Big|_{b_i=b_i^*} \tag{7}$$

$$= \frac{\mathrm{d}x_i(s_i(v_i))}{\mathrm{d}s_i(v_i)}\frac{\mathrm{d}s_i(v_i)}{\mathrm{d}v_i}\Big|_{s_i(v_i)=s_i^*(v_i)} \quad . \tag{8}$$

But now the first-order optimality conditions imply that the second term is necessarily zero, since $s_i^*(v_i)$ is a utility-maximizing bid. As the only non-zero term in Equation 5 is $x_i(s_i^*(v_i))$, the theorem is proved. $\square$

3 *Equilibrium Derivations via the Envelope Theorem*

Next, let's use the envelope theorem to analyze a symmetric first-price auction, meaning one in which the bidders' values are drawn i.i.d. from a bounded distribution F on $[\underline{v}, \overline{v}]$, for some $\underline{v} \leq \overline{v} \in \mathbb{R}$.

Assume a symmetric equilibrium $s^*(v)$ that is non-decreasing in v, so that a bidder with the highest value wins. At such an equilibrium, if the allocation probability is $x(v)$ and the winner pays her bid $s^*(v)$, each bidder's expected utility at equilibrium is given by $U^*(v) = x(v)(v - s^*(v))$. The probability that a bidder i with value v wins is the probability that $v \geq v_j$, for all $j \neq i$: i.e., $F^{n-1}(v)$. Each bidder's expected utility at equilibrium is thus:

$$U^*(v) = F^{n-1}(v)(v - s^*(v)) \quad . \tag{9}$$

By Theorem 2.1, $\frac{\mathrm{d}U^*(v)}{\mathrm{d}v} = x(s^*(v))$. Moreover, by the monotonicity assumption (i.e., $s^*(v)$ is non-decreasing in v), $x(s^*(v)) = F^{n-1}(v)$. Therefore, by the fundamental theorem of calculus,

$$U^*(v) = \int_{\underline{v}}^{v} F^{n-1}(t)\,\mathrm{d}t \quad , \tag{10}$$

as $U^*(\underline{v}) = 0$. Setting these two expressions for $U^*(v)$ (Equations 9 and 10) equal to one another yields

$$F^{n-1}(v)(v - s^*(v)) = \int_{\underline{v}}^{v} F^{n-1}(t)\,\mathrm{d}t \quad , \tag{11}$$

from which it follows that

$$s^*(v) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}(t)\,\mathrm{d}t}{F^{n-1}(v)} \quad . \tag{12}$$

Finally, since[3]

$$vF^{n-1}(v) - \int_{\underline{v}}^{v} F^{n-1}(t)\,\mathrm{d}t = \int_{\underline{v}}^{v} t\,\mathrm{d}F^{n-1} \quad , \tag{13}$$

[3] See Math'l Aside at the start of Lecture 5 on Myerson's optimal auction design.

it follows that

$$s^*(v) = \frac{\int_{\underline{v}}^{v} t \, \mathrm{d}F^{n-1}}{F^{n-1}(v)} \quad . \tag{14}$$

In other words, at equilibrium in a symmetric first-price auction, bidders shade their bids in such a way that the result is the expected bid of the bidder with the second-highest value, conditioned on their value being highest.

Remark 3.1. This derivation establishes *necessary* conditions for s^* to be a symmetric equilibrium of a symmetric first-price auction: i.e., if s^* is such an equilibrium, then it must take the form of Equation 12 (or equivalently, Equation 14).

Last week, we proved this same result in the special case of $F = U[0,1]$. We can easily recover last week's result from this week's as follows. First, since $\underline{v} = 0$ and $F^{n-1}(t) = t^{n-1}$,

$$\int_0^v F^{n-1}(t)dt = \int_0^v t^{n-1}dt = \frac{1}{n}t^n\Big|_0^{\overline{v}} = \frac{v^n}{n} \quad .$$

Second, plugging this calculation into Equation 12, and again using the fact that $F^{n-1}(v) = v^{n-1}$, yields:

$$s^*(v) = v - \frac{v^n}{nv^{n-1}} = v - \frac{v}{n} = v\left(1 - \frac{1}{n}\right) = \left(\frac{n-1}{n}\right)v \quad .$$

4 *Myerson's Payment Formula via the Envelope Theorem*

Finally, again using the envelope theorem, we prove (one direction of) Myerson's payment characterization theorem—that the DSIC assumption implies Myerson's payment formula.

Proof. Reverting back to our original notation, we denote bidder i's optimal utility (or value function) by V_i. By Theorem 2.1, $V_i'(v_i) = x_i(s_i^*(v_i))$. Moreover, by the DSIC assumption, bidder i's expected utility is maximized at v_i: i.e., $s_i^*(v_i) = v_i$. Therefore, $V_i'(v_i) = x_i(v_i)$.

More specifically, $V_i'(v_i, \mathbf{v}_{-i}) = x_i(v_i, \mathbf{v}_{-i})$. But then, by **??** (which invokes the fundamental theorem of calculus),

$$V_i(v_i, \mathbf{v}_{-i}) - V_i(\underline{v}_i, \mathbf{v}_{-i}) = \int_{\underline{v}_i}^{v_i} V_i'(z, \mathbf{v}_{-i}) \, \mathrm{d}z = \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z \quad .$$

Next, letting $p_i(v_i, \mathbf{v}_{-i})$ denote bidder i's *expected* payment, we can also express bidder i's expected utility $V_i(v_i, \mathbf{v}_{-i})$ as $v_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i})$.

It now follows that

$$v_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i}) - \left(\underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i})\right) = \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z \quad .$$

In other words,

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \int_{\underline{v}_i}^{v_i} x_i(v_i, \mathbf{v}_{-i}) \, \mathrm{d}z + p_i(\underline{v}_i, \mathbf{v}_{-i}) - \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) \quad .$$

□

References

[1] Dan Quint. Some beautiful theorems with beautiful proofs. University of Wisconsin–Madison, 2014.