Bayesian Games

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We describe incomplete-information, or Bayesian, normal-form games (formally; no examples), and corresponding equilibrium concepts.

1 *A Bayesian Model of Interaction*

A **Bayesian**, or **incomplete-information**, game is a generalization of a complete-information game. Recall that in a complete-information game, everything relevant to the play of the game is assumed to be common knowledge. In a Bayesian game, many things about the game are again common knowledge, but the players may have additional private information. This private information is captured by the notion of an **epistemic type**, which describes a player's private knowledge: i.e., all that is not common knowledge. The Bayesian-game formalism makes two simplifying assumptions:

- Any information that is not common knowledge pertains only to utilities. In particular, in all realizations of a Bayesian game, the number of players and each one's action set across realizations are identical.[1]
- Players maintain beliefs about the game (i.e., about utilities) in the form of a joint probability distribution over all players' types. Prior to receiving any private information, this probability distribution is common knowledge. As such, it is called a **common prior**.[2]

  After receiving private information, players condition on this newfound knowledge to update their beliefs. As a consequence of the common prior assumption, any differences in beliefs can be attributed entirely to differences in information.

[1] This is not a strong assumption. If this were not the case, e.g., if one player were unsure as to whether another player had one or two actions, a dummy action (e.g., a dominated strategy) could be added to a model of the game in the first case, so that the other player always has two actions.

[2] This *is* a strong assumption.

As in complete-information games, rational players are assumed to maximize their expected utilities. Further, they are assumed to update their beliefs via Bayes' rule when they learn new information.

Thus, in a Bayesian game, in addition to players, actions, and utilities, there is a type space $T = \prod_{i \in [n]} T_i$, where T_i is the type space of player i. There is also a common prior F, which is a probability distribution over type profiles. Utility functions then depend not only on actions, but on types as well: $u_i : A \times T \to \mathbb{R}$.

Again, as in complete-information games, it is assumed that all of the above is common knowledge among the players. Further, it is assumed that each player learns her own type: i.e., receives the relevant private information. An agent's strategy, then, becomes a function of its type: i.e., for all players i, $s_i : T_i \to A_i$. And a mixed strategy, as usual, is a probability distribution over (pure) strategies.[3]

[3] Mixed strategies in Bayesian games are complicated objects: they are probability distributions over functions!

2 Phases of a Bayesian Game

We can divide a Bayesian game into three phases:

- In the **ex-ante** phase, no player knows what her own type is, or the types of any other player. When we reason about a strategy in this phase, we use the following expected utility function:

$$\mathop{\mathbb{E}}_{\mathbf{t}\sim F}\left[u_i(s_i, \mathbf{s}_{-i}; t_i, \mathbf{t}_{-i})\right].$$

- In the **interim** phase, each player knows her own type, but not the types of any other player. When we reason about a strategy in this phase, we use the following expected utility function:

$$\mathop{\mathbb{E}}_{\mathbf{t}_{-i}\sim F_{\mathbf{t}_{-i}|t_i}}\left[u_i(s_i, \mathbf{s}_{-i}; t_i, \mathbf{t}_{-i})\right].$$

  The notation $F_{\mathbf{t}_{-i}|t_i}$ signifies the joint distribution over type profiles conditioned on i's private information.

- In the **ex-post** phase, each player knows her own type, and the types of every other player. When we reason about a strategy in this phase, we use the following utility function:

$$u_i(s_i, \mathbf{s}_{-i}; t_i, \mathbf{t}_{-i}).$$

  We summarize the phases in Figure 1:

Phase	Knows t_i?	Knows $\mathbf{t}_{-i}$?	Relevant Utility
Ex-ante	No	No	$\mathbb{E}_{\mathbf{t}\sim F}\left[u_i(s_i, \mathbf{s}_{-i}, t_i, \mathbf{t}_{-i})\right]$
Interim	Yes	No	$\mathbb{E}_{\mathbf{t}_{-i}\sim F_{\mathbf{t}_{-i}\mid t_i}}\left[u_i(s_i, \mathbf{s}_{-i}, t_i, \mathbf{t}_{-i})\right]$
Ex-post	Yes	Yes	$u_i(s_i, \mathbf{s}_{-i}, t_i, \mathbf{t}_{-i})$

Figure 1: A summary of the phases of a Bayesian Game.

3 Bayesian Equilibria in Bayesian Games

Corresponding to the three phases of a Bayesian game, there are three notions of equilibrium in a Bayesian game.

A strategy profile $\mathbf{s} = (s_i, \mathbf{s}_{-i}) \in S$ is an **ex-ante Bayes-Nash equilibrium** if no player can increase her ex-ante expected utility by unilaterally changing her strategy:

$$\mathop{\mathbb{E}}_{\mathbf{t}\sim F}\left[u_i(s_i(t_i), \mathbf{s}_{-i}(t_{-i}); t_i, \mathbf{t}_{-i})\right] \geq \mathop{\mathbb{E}}_{\mathbf{t}\sim F}\left[u_i(s_i'(t_i), \mathbf{s}_{-i}(t_{-i}); t_i, \mathbf{t}_{-i})\right], \quad \forall i \in [n], \forall s_i' \in S_i.$$

A strategy profile $\mathbf{s} = (s_i, \mathbf{s}_{-i}) \in S$ is an **interim Bayes-Nash equilibrium** if no player can increase her interim expected utility by unilaterally changing her strategy: $\forall i \in [n], \forall t_i \in T_i, \forall s_i' \in S_i$,

$$\mathop{\mathbb{E}}_{\mathbf{t}_{-i} \sim F_{\mathbf{t}_{-i}|t_i}} \left[u_i(s_i(t_i), \mathbf{s}_{-i}(t_{-i}); t_i, \mathbf{t}_{-i})\right] \geq \mathop{\mathbb{E}}_{\mathbf{t}_{-i} \sim F_{\mathbf{t}_{-i}|t_i}} \left[u_i(s_i'(t_i), \mathbf{s}_{-i}(t_{-i}); t_i, \mathbf{t}_{-i})\right].$$

Interestingly, ex-ante and interim Bayes-Nash equilibria turn out to be equivalent, in which case they are both referred to merely as **Bayes-Nash** equilibria (BNE). Moreover, as Bayes-Nash equilibria are Nash equilibria (imagine exploding a Bayesian game into a normal-form game; example forthcoming), Nash's theorem guarantees their existence, assuming T is finite.

4 *Ex-post Nash Equilibria in Bayesian Games*

A strategy profile $\mathbf{s} = (s_i, \mathbf{s}_{-i}) \in S$ is an **ex-post Nash equilibrium (EPNE)** if no player can increase her *ex-post* expected utility by unilaterally changing her strategy:

$$u_i(s_i(t_i), \mathbf{s}_{-i}(t_{-i}); t_i, \mathbf{t}_{-i}) \geq u_i(s_i'(t_i), \mathbf{s}_{-i}(t_{-i}); t_i, \mathbf{t}_{-i}), \quad \forall i \in [n], \forall s_i' \in S_i, \forall \mathbf{t} \in T.$$

EPNE is an equilibrium concept. As such, it assumes all players are best responding to one another: i.e., maximizing their utility, given the other players' strategies. It is a worst-case concept, however; it does not rely on the common prior assumption, and it does assume players are Bayesian, i.e., *expected* utility maximizers. Rather, each player's strategy, which is conditioned on her own type, must be a best response to the other players' strategies, which are likewise conditioned on their types, *regardless of one another's types*.

Taking this worst-case reasoning one step further, it is also possible to define DSE in incomplete-information games, by dropping both the common prior assumption and the assumption that players are utility maximizers! As in complete-information games, DSE need not exist in incomplete-information games (a strict generalization).

Formally, a strategy s_i for player $i \in [n]$ in an incomplete-information game is **dominant** if it is (weakly) optimal, regardless of the other players' actions *and types*: i.e.,

$$u_i(s_i(t_i), \mathbf{a}_{-i}; t_i, \mathbf{t}_{-i}) \geq u_i(s_i'(t_i), \mathbf{a}_{-i}; t_i, \mathbf{t}_{-i}), \quad \forall \mathbf{t} \in T, \forall s_i' \in S_i, \forall \mathbf{a}_{-i} \in A_{-i}$$

A strategy profile $\mathbf{s} \in S$ is a **dominant strategy equilibrium** (DSE) if all players play dominant strategies.

Like DSE, EPNE need not exist. To make this point, we present three examples, two games with EPNE, and a final game that combines the prior two, but does not have an EPNE.[4]

[4] This counterexample was borrowed from these lecture notes.

Example 4.1 (Example of an EPNE)**.** Consider a two-player Bayesian game with type space $T_1 = \{T\}$ and $T_2 = \{L, R\}$ and action spaces $A_1 = A_2 = \{C, D\}$. The payoffs of this game are described by the following two matrices, or subgames, each one corresponding to a possible type profile: i.e., TL and TR.

L (type profile T):

	C	D
C	2,2	0,0
D	3,0	1,1*

R:

	C	D
C	2,1	0,0
D	3,0	1,2*

Observe that these two subgames both have a pure-strategy Nash equilibrium profile, indicated by a *. Moreover, these equilibria are dominant (and hence, pure) strategy equilibria in their respective subgames, so they are the unique equilibria (mixed or pure) of these subgames. Consequently, in any ex-post equilibrium, players must play one of these two equilibrium profiles.

Player 2's actions in these equilibria cannot depend on player 1's type, as player 1 has only 1 possible type. So the interesting case to consider is player 1's actions; specifically, whether they vary with player 2's type. As they do not—player 1 plays D regardless of player 2's type—this game affords a unique EPNE, namely D, D.

Example 4.2 (Another example of an EPNE)**.** Consider a two-player Bayesian game with type space $T_1 = \{B\}$ and $T_2 = \{L, R\}$ and action spaces $A_1 = A_2 = \{C, D\}$. The payoffs of this game are described by the following two matrices, or subgames, each one corresponding to a possible type profile: i.e., BL and BR.

L (type profile B):

	C	D
C	1,2*	3,0
D	0,0	2,1

R:

	C	D
C	1,1*	3,0
D	0,0	2,2

As in the previous example, these two subgames both have a pure-strategy Nash equilibrium profile, indicated by a *. Moreover, these equilibria are dominant (and hence, pure) strategy equilibria in their respective subgames, so they are the unique equilibria (mixed or pure) of these subgames. Consequently, in any ex-post equilibrium, players must play one of these two equilibrium profiles.

Player 2's actions in these equilibria cannot depend on player 1's type, as player 1 has only 1 possible type. So the interesting case to consider is player 1's actions; specifically, whether they vary with player 2's type. As they do not—player 1 plays C regardless of player 2's type—this game affords a unique EPNE, namely C, C.

Example 4.3 (Counterexample to the existence of EPNE)**.** Consider a two-player Bayesian game with type spaces $T_1 = \{T, B\}$ and $T_2 =$

$\{L, R\}$ and action spaces $A_1 = A_2 = \{C, D\}$. The payoffs of this game are described by the following four matrices, or subgames, each one corresponding to a possible type profile: i.e., TL, TR, BL, or BR.

		L			R	
		C	D		C	D
T	C	2,2	0,0	C	2,1	0,0
	D	3,0	1,1*	D	3,0	1,2*
		C	D		C	D
B	C	1,2*	3,0	C	1,1*	3,0
	D	0,0	2,1	D	0,0	2,2

As above, each of these four subgames has a pure-strategy Nash equilibrium profile, indicated by a *. Moreover, these equilibria are dominant (and hence, pure) strategy equilibria in their respective subgames, so they are the unique equilibria (mixed or pure) of these subgames. Consequently, in any ex-post equilibrium, players must play one of these four equilibrium profiles.

The EPNE constraints do not pose a problem for player 1. When player 1 is of type T (as in Example 4.1), player 1 plays D in both equilibria, regardless of whether player 2 is of type L or R. Likewise, when player 1 is of type B (as in Example 4.2), player 1 plays C in both equilibria, regardless of whether player 2 is of type L or R.

But player 2 cannot satisfy the EPNE constraints. Regardless of player 2's type, she would have to play D when player 1 is of type T, and C when player 1 is of type B. As players cannot condition their play on *another* player's type, no EPNE exist in this game.