Bayesian Battle of the Sexes

CSCI 1440/2440

2025-01-29

We present an example of a Bayesian game. This set of notes is partially based on this video.

1 *Where Should Alice and Bob See a Movie Tonight?*

Two roommates, Alice and Bob, are planning to see a movie tonight, at one of two possible locations: the cinema (C), or at home (H).[1] Alice is interested in Bob, and would like to be in the same place as Bob, but she prefers to go to the cinema. Bob comes in two flavors; he may be Bob interested (I) or uninterested (U) in Alice, each with equal probability. If Bob turns out to also be interested in Alice, then he receives positive payoff for being with Alice, but prefers staying home to going to the cinema.[2] Conversely, if Bob turns out not to be interested in Alice, then he receives positive payoff for avoiding Alice, but still prefers staying home.

[1] And they don't communicate beforehand, because . . . *reasons*.

[2] COVID is still prevalent in RI, and he prefers not to have to mask up.

If Bob is interested in Alice, the utility Alice and Bob receive are given by Figure 1, where Alice is the row player, and Bob is the column player.

	C	H
C	10, 5	0, 0
H	0, 0	5, 10

$\Pr(I) = \frac{1}{2}$

Figure 1: The payoff matrix describing the payoffs Alice and Bob receive for attending C or H, if Bob is interested in Alice. Alice is the row player.

On the other and, if Bob is not interested in Alice, the utility Alice and Bob receive are given by Figure 2, where Alice is again the row player, and Bob, the column player.

	C	H
C	10, 0	0, 10
H	0, 5	5, 0

$\Pr(U) = \frac{1}{2}$

Figure 2: The payoff matrix describing the payoffs Alice and Bob receive for attending C or H, if Bob is *not* interested in Alice. Alice is the row player.

We take as our common prior the uniform distribution $F = (1/2, 1/2)$.

2 *Representing this Bayesian Game in the Normal Form*

The ex-ante expected utility of player i assuming strategy profile $\mathbf{s}$ is:

$$\mathbb{E}\left[u_i(\mathbf{s})\right] = \mathop{\mathbb{E}}_{\mathbf{t}\sim F}\left[u_i(\mathbf{s};\mathbf{t})\right]. \tag{1}$$

Using this formula, we can describe ex-ante expected utilities for any strategy profile in our Bayesian game, which leads to a representation of this game as a complete-information normal form game, as hinted at in Figure 3, complete with expected utilities in Figure 4.

	CC	CH	HC	HH
C				
H				

Figure 3: The expected payoffs in Bayesian Battle of the Sexes.

For example, suppose Alice plays C, and Bob plays C if he has type I, and H if he has type U. We use the notation CH as shorthand to describe Bob's strategy, where the first letter C indicates Bob's action when his type is I, and the second letter H indicates his action when his type is U.

Now Alice's expected utility is:

$$\begin{aligned}
\mathbb{E}\left[u_A(C,CH)\right] &= \sum_{\mathbf{t}\in T} \Pr(\mathbf{t})\, u_A(C(t_a),CH(t_b);\mathbf{t}) && (2)\\
&= \Pr(I)\, u_A(C,CH(I);I) + \Pr(U)\, u_A(C,CH(U);U) && (3)\\
&= \Pr(I)\, u_A(C,C;I) + \Pr(U)\, u_A(C,H;U) && (4)\\
&= \frac{1}{2}(10) + \frac{1}{2}(0) && (5)\\
&= 5. && (6)
\end{aligned}$$

And, Bob's expected utility is:

$$\begin{aligned}
\mathbb{E}\left[u_B(CH,C)\right] &= \sum_{\mathbf{t}\in T} \Pr(\mathbf{t})\, u_B(CH(t_b),C(t_a);\mathbf{t}) && (7)\\
&= \Pr(I)\, u_B(CH(I),C;I) + \Pr(U)\, u_B(CH(U),C;U) && (8)\\
&= \Pr(I)\, u_B(C,C;I) + \Pr(U)\, u_B(H,C;U) && (9)\\
&= \frac{1}{2}(5) + \frac{1}{2}(10) && (10)\\
&= \frac{15}{2}. && (11)
\end{aligned}$$

We can continue in this fashion to compute all the ex-ante expected utilities in this Bayesian version of Battle of the Sexes, which yields the following normal-form representation of the Bayesian game:

There are no dominated strategies in this game. Still, this game has one pure-strategy Nash equilibria, (C, CH), indicated by a *, which leads to payoffs of 5 for Alice and 15/2 for Bob.

	CC	CH	HC	HH
C	(10, 5/2)	(5, 15/2)*	(5, 0)	(0, 5)
H	(0, 5/2)	(5/2, 0)	(5/2, 15/2)	(5, 0)

Figure 4: The expected payoffs in Bayesian Battle of the Sexes.

3 *Finding Mixed Strategy Nash Equilibria*

In addition to the one pure-strategy Nash equilibrium, there, are potentially more equilibria, namely mixed-strategy Nash equilibria.

- Let p be the probability that Alice plays C.
- Let q_I be the probability that Bob plays C, if Bob is interested in Alice.
- Let q_U be the probability that Bob plays C, if Bob is uninterested in Alice.

If Alice uses the mixed strategy p, how should Bob respond? Well, this depends on whether Bob has type I or type U. We will derive two mixed-strategy Nash equilibria, each one corresponding to whether we start by assuming Bob has type I or type U.

3.1 *Starting Point: If Bob is Interested*

If Bob is interested in Alice, his payoff for playing C is:

$$5p + 0(1 - p) = 5p. \tag{12}$$

His payoff for playing H is:

$$0p + 10(1 - p) = 10 - 10p. \tag{13}$$

What value of p for Alice makes interested Bob indifferent between his two actions? Equating the payoffs, and solving for p yields:

$$5p = 10 - 10p \tag{14}$$

$$15p = 10 \tag{15}$$

$$p = \frac{2}{3} \tag{16}$$

Therefore, the mixed strategy $\left(\frac{2}{3}, \frac{1}{3}\right)$ for Alice makes interested Bob indifferent between his two actions.

Let's assume Alice playes this mixed strategy. If Bob is uninterested in Alice, his payoff for playing C is:

$$0p + 5(1 - p) = 5(1 - p) \tag{17}$$

His payoff for playing H is:

$$10p + 0(1 - p) = 10p. \tag{18}$$

Plugging in Alice's mixed strategy yields a payoff of $5 - 5\left(\frac{2}{3}\right) = \frac{5}{3}$ for playing C, and $10\left(\frac{2}{3}\right) = \frac{20}{3}$ for playing H. Uninterested Bob's payoff is strictly greater when playing H, so $q_U = 0$, meaning he plays H.

So let's assume uninterested Bob plays H. What should interested Bob do? Alice's payoff for playing C, when $q_U = 0$, is:

$$\Pr(I)\left[10q_I + 0(1-q_I)\right] + \Pr(U)\left[10q_U + 0(1-q_U)\right] \tag{19}$$

$$= \frac{1}{2}\left[10q_I\right] + \frac{1}{2}\left[10q_U\right] \tag{20}$$

$$= \frac{1}{2}\left[10q_I\right] + \frac{1}{2}\left[10(0)\right] \tag{21}$$

$$= 5q_I. \tag{22}$$

Her payoff for playing H, when $q_U = 0$, is:

$$\Pr(I)\left[0q_I + 5(1-q_I)\right] + \Pr(U)\left[0q_U + 5(1-q_U)\right] \tag{23}$$

$$= \frac{1}{2}\left[5(1-q_I)\right] + \frac{1}{2}\left[5(1-q_U)\right] \tag{24}$$

$$= \frac{1}{2}\left[5(1-q_I)\right] + \frac{1}{2}\left[5(1)\right] \tag{25}$$

$$= \frac{5}{2}(1-q_I) + \frac{5}{2}. \tag{26}$$

What value of q_I for interested Bob makes Alice indifferent between her two actions? Equating the payoffs, and solving for q_I yields:

$$5q_I = \frac{5}{2}(1-q_I) + \frac{5}{2} \tag{27}$$

$$10q_I = 5(1-q_I) + 5 \tag{28}$$

$$15q_I = 10 \tag{29}$$

$$q_I = \frac{2}{3} \tag{30}$$

Thus, interested Bob makes Alice indifferent by playing $\left(\frac{2}{3}, \frac{1}{3}\right)$.

Putting it all together:

1. Alice plays C with probability $p = \frac{2}{3}$, and H with probability $1 - p = \frac{1}{3}$.
2. If Bob has type I, then he plays C with probability $q_I = \frac{2}{3}$, and H with probability $1 - q_I = \frac{1}{3}$.
3. If Bob has type U, then he plays C with probability $q_U = 0$, and H with probability $1 - q_U = 1$.

We summarize this mixed strategy as follows:

$$\left(\underbrace{\left(\frac{2}{3}, \frac{1}{3}\right)}_{\text{Alice}}, \underbrace{\left(\underbrace{\left(\frac{2}{3}, \frac{1}{3}\right)}_{\text{Type I}}, \underbrace{(0,1)}_{\text{Type U}} \right)}_{\text{Bob}} \right). \tag{31}$$

So Bob plays strategy CH with probability 2/3, and strategy HH with probability 1/3. Mixing in Alice's strategy yields the following joint distribution over Alice's and Bob's strategies at this equilibrium:

	CC	CH	HC	HH
C	0	4/9	0	2/9
H	0	2/9	0	1/9

Figure 5: The joint probabilities at this mixed-strategy Nash equilibirum.

At this equilibrium, Alice's utility is 30/9 and Bob's is 35/9.

Summary To summarize, our solution strategy was as follows:

1. Assume Alice plays a mixed strategy, and derive the precise mixture that makes interested Bob indifferent between his two actions, so that interested Bob plays a mixed strategy.

2. Assume Alice plays this precise mixed strategy, and derive a best response for uninterested Bob.

3. Assume uninterested Bob plays this best response, and derive a mixed strategy for interested Bob that makes Alice indifferent between her two actions.

Verification We now verify that this mixed strategy is in fact a Bayes-Nash equilibrium. Fixing Alice's (Bob's) strategy, it should be the case that Bob (Alice) cannot employ an alternative mixed strategy that yields strictly more utility.

Alice The expected utility Alice receives for playing C is:

$$u_A(C) = \Pr(I)\,[10\,(q_I) + 0\,(1-q_I)] + \Pr(U)\,[10\,(q_U) + 0\,(1-q_U)] \tag{32}$$

$$= \frac{1}{2}\left[10\left(\frac{2}{3}\right) + 0\left(\frac{1}{3}\right)\right] + \frac{1}{2}\,[10\,(0) + 0\,(1-0)] \tag{33}$$

$$= \frac{10}{3}. \tag{34}$$

The expected utility Alice receives for playing H is:

$$u_A(H) = \Pr(I)\left[0\left(q_I\right) + 5\left(1 - q_I\right)\right] + \Pr(U)\left[0\left(q_U\right) + 5\left(1 - q_U\right)\right] \tag{35}$$

$$= \frac{1}{2}\left[0\left(\frac{2}{3}\right) + 5\left(\frac{1}{3}\right)\right] + \frac{1}{2}\left[0\left(0\right) + 5\left(1 - 0\right)\right] \tag{36}$$

$$= \frac{10}{3}. \tag{37}$$

Since the expected utilities are equal, Alice is indifferent between playing C and H, and cannot improve her expected utility.

Bob (type I) The expected utility Bob (type I) receives for playing C is:

$$u_B(C) = 5\left(p\right) + 0\left(1 - p\right) \tag{38}$$

$$= 5p \tag{39}$$

$$= \frac{10}{3}. \tag{40}$$

The expected utility Bob (type I) receives for playing H is:

$$u_B(H) = 0\left(p\right) + 10\left(1 - p\right) \tag{41}$$

$$= 10\left(1 - p\right) \tag{42}$$

$$= \frac{10}{3}. \tag{43}$$

Since the expected utilities are equal, Bob is indifferent between playing C and H, and cannot improve his expected utility.

Bob (type U) The expected utility Bob (type U) receives for playing C is:

$$u_B(C) = 0\left(p\right) + 5\left(1 - p\right) \tag{44}$$

$$= 5(1 - p) \tag{45}$$

$$= \frac{5}{3}. \tag{46}$$

The expected utility Bob (type U) receives for playing H is:

$$u_B(H) = 10\left(p\right) + 0\left(1 - p\right) \tag{47}$$

$$= 10\left(p\right) \tag{48}$$

$$= \frac{20}{3}. \tag{49}$$

Since the expected utility of playing H is strictly larger than the expected utility of playing C, Bob will play H.

3.2 Starting Point: If Bob is Uninterested

If Bob is uninterested in Alice, his payoff for playing C is:

$$0p + 5(1-p) = 5(1-p). \tag{50}$$

His payoff for playing H is:

$$10p + 0(1-p) = 10p. \tag{51}$$

What value of p for Alice makes uninterested Bob indifferent between his two actions? Equating the payoffs, and solving for p yields:

$$5 - 5p = 10p \tag{52}$$
$$5 = 15p \tag{53}$$
$$p = \frac{1}{3} \tag{54}$$

Therefore, the mixed strategy $\left(\frac{1}{3}, \frac{2}{3}\right)$ for Alice makes uninterested Bob indifferent between his two actions.

Let's assume Alice plays this mixed strategy. If Bob is interested in Alice, his payoff for playing C is:

$$5p + 0(1-p) = 5p. \tag{55}$$

His payoff for playing H is

$$0p + 10(1-p) = 10(1-p). \tag{56}$$

Plugging in Alice's mixed strategy yields a payoff of $5\left(\frac{1}{3}\right) = \frac{5}{3}$ for playing C, and $10\left(\frac{2}{3}\right) = \frac{20}{3}$ for playing H. Interested Bob's payoff is strictly greater when playing H, so $q_I = 0$, meaning he plays H.

So let's assume interested Bob plays H. What should uninterested Bob do? Alice's payoff Alice for playing C, when $q_I = 0$, is:

$$\Pr(I)\left[10q_I + 0(1-q_I)\right] + \Pr(U)\left[10q_U + 0(1-q_U)\right] \tag{57}$$
$$= \frac{1}{2}\left[10q_I\right] + \frac{1}{2}\left[10q_U\right] \tag{58}$$
$$= \frac{1}{2}\left[10(0)\right] + \frac{1}{2}\left[10q_U\right] \tag{59}$$
$$= 5q_U. \tag{60}$$

Her payoff for playing H, when $q_I = 0$, is:

$$\Pr(I)\left[0q_I + 5(1-q_I)\right] + \Pr(U)\left[0q_U + 5(1-q_U)\right] \tag{61}$$
$$= \frac{1}{2}\left[5(1-q_I)\right] + \frac{1}{2}\left[5(1-q_U)\right] \tag{62}$$

$$= \frac{1}{2}\left[5(1)\right] + \frac{1}{2}\left[5(1-q_U)\right] \tag{63}$$

$$= \frac{5}{2} + \frac{5}{2}(1-q_U). \tag{64}$$

What value of q_U for uninterested Bob makes Alice indifferent between her two actions? Equating the payoffs, we solve for q_U:

$$5q_U = \frac{5}{2} + \frac{5}{2}(1-q_U) \tag{65}$$

$$10q_U = 5 + 5(1-q_U) \tag{66}$$

$$15q_U = 10 \tag{67}$$

$$q_U = \frac{2}{3} \tag{68}$$

Thus, uninterested Bob makes Alice indifferent by playing $\left(\frac{2}{3}, \frac{1}{3}\right)$.

Putting it all together:

1. Alice plays C with probability $p = \frac{1}{3}$, and H with probability $1 - p = \frac{2}{3}$.
2. If Bob has type I, then he plays C with probability $q_I = 0$, and H with probability $1 - q_I = 1$.
3. If Bob has type U, then he plays C with probability $q_U = \frac{2}{3}$, and H with probability $1 - q_U = \frac{1}{3}$.

We summarize this mixed strategy as follows:

$$\left(\underbrace{\left(\frac{1}{3}, \frac{2}{3}\right)}_{\text{Alice}}, \underbrace{\left(\underbrace{(0,1)}_{\text{Type I}} \underbrace{\left(\frac{2}{3}, \frac{1}{3}\right)}_{\text{Type U}}, \right)}_{\text{Bob}} \right). \tag{69}$$

So Bob plays strategy HC with probability 2/3, and strategy HH with probability 1/3. Mixing in Alice's strategy yields the following joint distribution over Alice's and Bob's strategies at this equilibrium:

	CC	CH	HC	HH
C	0	0	2/9	1/9
H	0	0	4/9	2/9

Figure 6: The joint probabilities at this mixed-strategy Nash equilibirum.

At this equilibrium, Alice's utility is 30/9 and Bob's is 35/9.

Summary To summarize, our solution strategy was as follows:

1. Assume Alice plays a mixed strategy, and derive the precise mixture that makes uninterested Bob indifferent between his two actions, so that uninterested Bob plays a mixed strategy.
2. Assume Alice plays this precise mixed strategy, and derive a best response for interested Bob.
3. Assume interested Bob plays this best response, and derive a mixed strategy for uninterested Bob that makes Alice indifferent between her two actions.

Verification We now verify that this mixed strategy is in fact a Bayes-Nash equilibrium. Fixing Alice's (Bob's) strategy, it should be the case that Bob (Alice) cannot employ an alternative mixed strategy that yields strictly more utility.

Alice The expected utility Alice receives for playing C is

$$u_A(C) = \Pr(I)\,[10\,(q_I) + 0\,(1-q_I)] + \Pr(U)\,[10\,(q_U) + 0\,(1-q_U)] \tag{70}$$

$$= \frac{1}{2}\,[10\,(0) + 0\,(1)] + \frac{1}{2}\left[10\left(\frac{2}{3}\right) + 0\left(\frac{1}{3}\right)\right] \tag{71}$$

$$= \frac{10}{3}. \tag{72}$$

The expected utility Alice receives for playing H is

$$u_A(H) = \Pr(I)\,[0\,(q_I) + 5\,(1-q_I)] + \Pr(U)\,[0\,(q_U) + 5\,(1-q_U)] \tag{73}$$

$$= \frac{1}{2}\,[0\,(0) + 5\,(1)] + \frac{1}{2}\left[0\left(\frac{2}{3}\right) + 5\left(\frac{1}{3}\right)\right] \tag{74}$$

$$= \frac{10}{3}. \tag{75}$$

Since the expected utilities are equal, Alice is indifferent between playing C and H, and cannot improve her expected utility.

Bob (type I) The expected utility Bob (type I) receives for playing C is

$$u_B(C) = 5\,(p) + 0\,(1-p) \tag{76}$$

$$= 5p \tag{77}$$

$$= \frac{5}{3}. \tag{78}$$

The expected utility Bob (type I) receives for playing H is

$$u_B(H) = 0\,(p) + 10\,(1-p) \tag{79}$$
$$= 10\,(1-p) \tag{80}$$
$$= \frac{20}{3}. \tag{81}$$

Since the expected utility of playing H is strictly larger than the expected utility of playing C, Bob will play H.

Bob (type U) The expected utility Bob (type U) receives for playing C is

$$u_B(C) = 0\,(p) + 5\,(1-p) \tag{82}$$
$$= 5(1-p) \tag{83}$$
$$= \frac{10}{3}. \tag{84}$$

The expected utility Bob (type U) receives for playing H is

$$u_B(H) = 10\,(p) + 0\,(1-p) \tag{85}$$
$$= 10\,(p) \tag{86}$$
$$= \frac{10}{3}. \tag{87}$$

Since the expected utilities are equal, Bob is indifferent between playing C and H, and cannot improve his expected utility.