Battle of the Sexes

CSCI 1440/2440

2025-01-22

We present an example of a (complete-information) normal-form games, and calculate its Nash equilibria.

1   Battle of the Sexes

Alice and Bob are planning to attend an event together this evening. Unfortunately, both are forgetful, and can't recall which of two possible events they had planned to attend. To make matters worse, they cannot communicate with each other before the event.[1] While each person prefers one event over the other, both Alice and Bob derive no happiness if they don't attend the same event. Payoffs for all possible outcomes are given in Figure 1.

[1] This thought experiment was created before the proliferation of cellular technology.

	Concert	Lecture
Concert	7, 3	0, 0
Lecture	0, 0	3, 7

Figure 1: The payoff matrix describing the payoffs Alice and Bob receive for attending one of two possible events. Alice is the row player. Bob is the column player. If Alice and Bob both attend the Concert, Alice receives payoff 7, and Bob receives payoff 3.

Problem

1. Draw the best-response correspondences for each player on the same graph.[2]

2. Label all Nash equilibria that involve pure strategies.

3. Label all Nash equilibria that involve mixed strategies.

4. What are the players' expected utilities (a.k.a payoffs) at each of these Nash equilibria?

[2] A correspondence is a generalization of a function in which an element of the domain may map to a subset (rather than just one element) of the co-domain.

Notation

We use the following notation:

- p: Probability Alice goes to the concert.
  $\boldsymbol{p} = (p, 1 - p)$ is Alice's mixed strategy.

- q: Probability Bob goes to the concert.
  $\boldsymbol{q} = (q, 1 - q)$ is Bob's mixed strategy.

- u_x: utility of $x \in \{A, B\}$, where A stands for Alice, and B stands for Bob.

2 Solution

By assumption, both Alice and Bob are expected utility maximizers.[3] In this game, this means that Alice seeks to maximize $u_A(\boldsymbol{p}, \boldsymbol{q}) \doteq \mathbb{E}_{\boldsymbol{p}}[u_A(\cdot, \boldsymbol{q})]$, while Bob seeks to maximize $u_B(\boldsymbol{p}, \boldsymbol{q}) \doteq \mathbb{E}_{\boldsymbol{q}}[u_B(\boldsymbol{p}, \cdot)]$.

We expand Alice's expected utility $u_A(\boldsymbol{p}, \boldsymbol{q})$ as follows:

$$\mathbb{E}_{\boldsymbol{p}}[u_A(\cdot, \boldsymbol{q})] = p \, \mathbb{E}_{\boldsymbol{q}}[u_A(\text{concert}, \cdot)] + (1-p) \, \mathbb{E}_{\boldsymbol{q}}[u_A(\text{lecture}, \cdot)] \qquad (1)$$

Then, to maximize Alice's expected utility, we take the derivative of Equation 1 w.r.t. p, Alice's decision variable. This derivative is

$$\mathbb{E}_{\boldsymbol{q}}[u_A(\text{concert}, \cdot)] - \mathbb{E}_{\boldsymbol{q}}[u_A(\text{lecture}, \cdot)] \qquad (2)$$

Setting this derivative equal to zero yields Alice's (necessary) condition for optimality, namely $\mathbb{E}_{\boldsymbol{q}}[u_A(\text{concert}, \cdot)] = \mathbb{E}_{\boldsymbol{q}}[u_A(\text{lecture}, \cdot)]$. This condition states that regardless of her strategy p, Alice's expected utility is maximized whenever Bob's strategy ensures that her expected utility for concert equals her expected utility for lecture.

Bob's reasoning is symmetric. His necessary condition for optimality, namely $\mathbb{E}_{\boldsymbol{p}}[u_B(\cdot, \text{concert})] = \mathbb{E}_{\boldsymbol{p}}[u_B(\cdot, \text{lecture})]$, states that regardless of his strategy q, Bob's expected utility is maximized whenever Alice's strategy ensures that his expected utility for concert equals his expected utility for lecture.

2.1 Alice

Alice prefers the concert when her expected utility of the concert action exceeds that of the lecture action. She prefers the lecture when her expected utility of the lecture action exceeds that of the concert action. When these two expected utilities are equal, she is indifferent between her two pure strategies, and willing to mix.

More formally, if $\mathbb{E}_{\boldsymbol{q}}[u_A(\text{concert}, \cdot)] > \mathbb{E}_{\boldsymbol{q}}[u_A(\text{lecture}, \cdot)]$, then $p = 1$ is optimal, and Alice goes to the concert. If $\mathbb{E}_{\boldsymbol{q}}[u_A(\text{lecture}, \cdot)] > \mathbb{E}_{\boldsymbol{q}}[u_A(\text{concert}, \cdot)]$, then then $p = 0$ is optimal, and Alice goes to the lecture. Otherwise, Alice is indifferent, and $p \in (0, 1)$. To summarize:

$$p \begin{cases} = 1, & \mathbb{E}_{\boldsymbol{q}}[u_A(\text{concert}, \cdot)] > \mathbb{E}_{\boldsymbol{q}}[u_A(\text{lecture}, \cdot)] \\ = 0, & \mathbb{E}_{\boldsymbol{q}}[u_A(\text{lecture}, \cdot)] > \mathbb{E}_{\boldsymbol{q}}[u_A(\text{concert}, \cdot)] \\ \in [0, 1], & \mathbb{E}_{\boldsymbol{q}}[u_A(\text{lecture}, \cdot)] = \mathbb{E}_{\boldsymbol{q}}[u_A(\text{concert}, \cdot)]. \end{cases} \qquad (3)$$

Next, let's calculate these expected utilities. Alice's expected utility of going to the concert is:

$$\mathbb{E}_{\boldsymbol{q}}[u_A(\text{concert}, \cdot)] = q u_A(\text{concert}, \text{concert}) + (1-q) u_A(\text{concert}, \text{lecture})$$

$$= q u_A(\text{concert}, \text{concert}). \qquad (4)$$

Alice's expected utility of going to the lecture is:

$$\mathbb{E}_q\left[u_A(\text{lecture}, \cdot)\right] = q u_A(\text{lecture}, \text{concert}) + (1-q)u_A(\text{lecture}, \text{lecture})$$

$$= (1-q)u_A(\text{lecture}, \text{lecture}). \tag{5}$$

Plugging in the numbers from the payoff matrix:

$$\mathbb{E}_q\left[u_A(\text{concert}, \cdot)\right] = \mathbb{E}_q\left[u_A(\text{lecture}, \cdot)\right] \tag{6}$$

$$q u_A(\text{concert}, \text{concert}) = (1-q)u_A(\text{lecture}, \text{lecture}) \tag{7}$$

$$7q = 3(1-q) \tag{8}$$

$$10q = 3 \tag{9}$$

$$q = \frac{3}{10} \tag{10}$$

Hence,

$$p \begin{cases} = 1, & \text{if } q > \frac{3}{10} \\ = 0, & \text{if } q < \frac{3}{10} \\ \in [0,1], & \text{if } q = \frac{3}{10}. \end{cases} \tag{11}$$

Alice's **best-response correspondence** is depicted in Figure 2. Note that this correspondence is *not* a function.
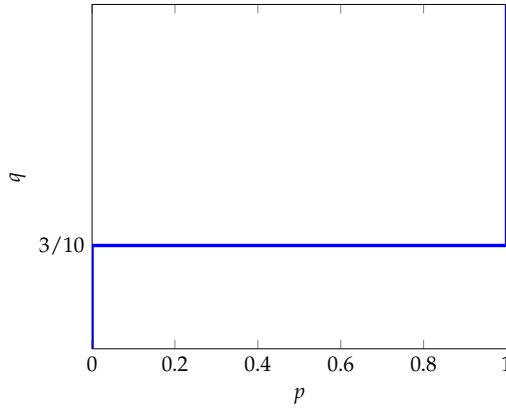

Figure 2: Alice's best response correspondence.

2.2 Bob

Like Alice, Bob prefers the concert when his expected utility of the concert action exceeds that of the lecture action. He prefers the lecture when his expected utility of the lecture action exceeds that of the concert action. When these two expected utilities are equal, he is indifferent between his two pure strategies, and willing to mix.

More formally, if $\mathbb{E}_p\left[u_B(\text{concert}, \cdot)\right] > \mathbb{E}_p\left[u_B(\text{lecture}, \cdot)\right]$, then $q = 1$ is optimal, and Bob goes to the concert. If $\mathbb{E}_p\left[u_B(\text{lecture}, \cdot)\right] >$

$\mathbb{E}_p[u_B(\text{concert}, \cdot)]$, then $q = 0$ is optimal, and Bob goes to the lecture. Otherwise, Bob is indifferent, and $q \in (0, 1)$. To summarize:

$$q \begin{cases} = 1, & \mathbb{E}_p[u_B(\text{concert}, \cdot)] > \mathbb{E}_p[u_B(\text{lecture}, \cdot)] \\ = 0, & \mathbb{E}_p[u_B(\text{lecture}, \cdot)] > \mathbb{E}_p[u_B(\text{concert}, \cdot)] \\ \in [0, 1], & \mathbb{E}_p[u_B(\text{lecture}, \cdot)] = \mathbb{E}_p[u_B(\text{concert}, \cdot)]. \end{cases} \quad (12)$$

Next, let's calculate these expected utilities. Bob's expected utility of going to the concert is:

$$\mathbb{E}_p[u_B(\text{concert}, \cdot)] = p u_B(\text{concert}, \text{concert}) + (1 - p) u_B(\text{concert}, \text{lecture})$$
$$= p u_B(\text{concert}, \text{concert}). \quad (13)$$

Bob's expected utility of going to the lecture is:

$$\mathbb{E}_p[u_B(\text{lecture}, \cdot)] = p u_B(\text{lecture}, \text{concert}) + (1 - p) u_B(\text{lecture}, \text{lecture})$$
$$= (1 - p) u_B(\text{lecture}, \text{lecture}). \quad (14)$$

Plugging in the numbers from the payoff matrix:

$$\mathbb{E}_p[u_B(\text{concert}, \cdot)] = \mathbb{E}_p[u_B(\text{lecture}, \cdot)] \quad (15)$$
$$p u_B(\text{concert}, \text{concert}) = (1 - p) u_B(\text{lecture}, \text{lecture}) \quad (16)$$
$$3p = 7(1 - p) \quad (17)$$
$$10p = 7 \quad (18)$$
$$p = \frac{7}{10} \quad (19)$$

Hence,

$$q \begin{cases} = 1, & \text{if } p > \frac{7}{10} \\ = 0, & \text{if } p < \frac{7}{10} \\ \in [0, 1], & p = \frac{7}{10}. \end{cases} \quad (20)$$

Bob's **best-response correspondence** is depicted in Figure 3. Note that this correspondence is *not* a function.

2.3   *Nash Equilibria*

Figure 4 plots the two curves together. From these overlapping plots, we can visualize the equilibrium solutions. They occur at the intersections of the two best-respondence correspondences.

2.4   *Utility*

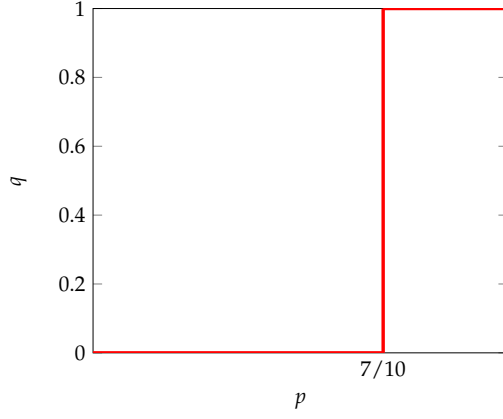We now compute players' expected utilities at the mixed strategy Nash equilibrium.
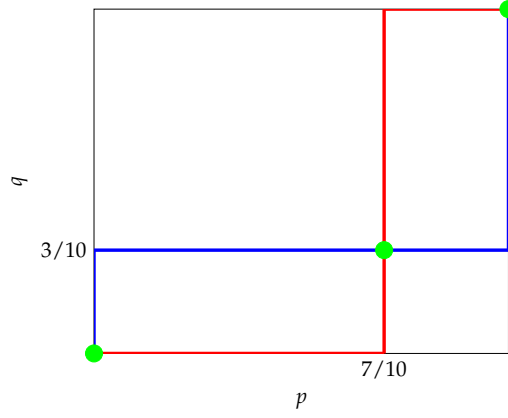
Figure 3: Bob's best response correspondence.

Figure 4: Both Alice's and Bob's best response correspondences.

Alice's expected utility $u_A(p, q)$ is:

$$\underset{p,q}{\mathbb{E}}\left[u_A(\cdot, \cdot)\right] = p \underset{q}{\mathbb{E}}\left[u_A(\text{concert}, \cdot)\right] + (1 - p) \underset{q}{\mathbb{E}}\left[u_A(\text{lecture}, \cdot)\right] \tag{21}$$

$$= pq u_A(\text{concert}, \text{concert}) + (1 - p)(1 - q) u_A(\text{lecture}, \text{lecture}) \tag{22}$$

$$= \frac{7}{10} \cdot \frac{3}{10} \cdot 7 + \frac{3}{10} \cdot \frac{7}{10} \cdot 3 \tag{23}$$

$$= \frac{21}{10} \tag{24}$$

Bob's expected utility $u_B(p, q)$ is:

$$\underset{p,q}{\mathbb{E}}\left[u_B(\cdot, \cdot)\right] = q \underset{p}{\mathbb{E}}\left[u_B(\text{concert}, \cdot)\right] + (1 - q) \underset{p}{\mathbb{E}}\left[u_B(\text{lecture}, \cdot)\right] \tag{25}$$

$$= qp u_B(\text{concert}, \text{concert}) + (1 - q)(1 - p) u_B(\text{lecture}, \text{lecture}). \tag{26}$$

Since the game is symmetric, $\mathbb{E}_q\left[u_B(\cdot, \cdot)\right] = \frac{21}{10}$ as well.