The Winner Determination Problem

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The *winner determination* problem is the optimization problem at the core of the VCG mechanism: finding an economically efficient, or welfare-maximizing, allocation of goods to buyers.

1 *The Primal*

Implementing VCG requires solving the winner determination (WD) problem (i.e., solving for an optimal allocation) $n+1$ times. This is a serious impediment to its widespread use, because solving the winner determination problem turns out to be an NP-hard problem. Nonetheless, as it is a problem of great practical importance, it has been studied empirically,[1] like other practical NP-hard problems: e.g., satisfiability and the travelling salesperson problem.

[1] Kevin Leyton-Brown, Mark Pearson, and Yoav Shoham. Towards a universal test suite for combinatorial auction algorithms. In *Proceedings of the 2nd ACM conference on Electronic commerce*, pages 66–76. ACM, 2000

WD can be formulated as a constrained optimization problem, where the objective is to maximize welfare, and the constraints are:

1. Do not allocate any good to more than one bidder.
2. Do not allocate to any bidder more than one bundle.[2]

[2] This set of constraints is necessary because the value of bundle $S_1 \cup S_2$ may not equal the value of S_1 plus the value of S_2, and maximizing the sum total of these values is the objective.

Let $x_{i,S}$ indicate whether bidder i is allocated bundle S or not:

$$x_{i,S} = \begin{cases} 1, & \text{if } i \text{ is allocated bundle } S \\ 0, & \text{otherwise.} \end{cases}$$

Finding an optimal allocation can then be expressed as an integer linear program (ILP). Given truthful reports $\mathbf{v}$, the primal is:

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i \in [n]} \sum_{S \subseteq G} x_{i,S} v_i(S) \\ \text{subject to} \quad & \sum_{i \in [n]} \sum_{S \subseteq G : j \in S} x_{i,S} \leq 1, && \forall j \in G \\ & \sum_{S \subseteq G} x_{i,S} \leq 1, && \forall i \in [n] \\ & x_{i,S} \in \{0,1\}, && \forall i \in [n], \forall S \subseteq G. \end{aligned}$$

This problem is computationally expensive because there are an exponential number of variables, but it would be difficult to solve regardless, because integer linear programming is NP-hard.

Remark 1.1. Not all instances of the winner determination problem are NP-hard, even in the multiparameter case. For example, when bidders' valuations are additive, so that the value of a set of goods is the sum total of their individual values: i.e., $v_i(S) = \sum_{j \in S} v_i(j)$, for all $i \in [n]$, the multiparameter case reduces to a single-parameter k-good case, and winner determination can be solved in polynomial time.

2 *The Dual*

Now, suppose we relax the integrality constraints in the ILP to arrive at LP in which allocations are now in $[0,1]$ instead of $\{0,1\}$. With a bit of work, we can show that the dual of this relaxed LP is:

$$\begin{aligned} \min_{\mathbf{u},\mathbf{p}} \quad & \sum_{i \in [n]} u_i + \sum_{j \in G} p_j \\ \text{subject to} \quad & u_i \geq v_i(S) - \sum_{j \in S} p_j, && \forall i \in [n], \forall S \subseteq G \\ & u_i \geq 0, && \forall i \in [n] \\ & p_j \geq 0, && \forall j \in G. \end{aligned}$$

We have intentionally used the variable names u and p, as the dual can be interpreted as solving for utilities and payments, respectively. The last two sets of constraints state that these quantities must be non-negative. The first set of constraints states that each bidder's utility must exceed its **indirect utility**, namely $\max_{S \subseteq G} v_i(S) - \sum_{j \in S} p_j$. In other words, no bidder can obtain higher utility at prices $\mathbf{p}$ than u_i. Additionally, the dual objective is the same as that of the primal, namely social welfare: in the primal, the goal is to maximize welfare, expressed in terms of values, whereas in the dual, the goal is to minimize the sum of utilities and prices (in other words, welfare). Finally, while the number of variables in the dual is only polynomial in size, the number of constraints is exponential.

A *The Primal and the Dual*

In matrix form, given $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{y} \in \mathbb{R}^m$, the relaxed primal can be written as

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

In matrix form, the dual can be written as

$$\begin{aligned} \min_{\mathbf{y}} \quad & \mathbf{b}^T \mathbf{y} \\ \text{subject to} \quad & A^T \mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

References

[1] Kevin Leyton-Brown, Mark Pearson, and Yoav Shoham. Towards a universal test suite for combinatorial auction algorithms. In *Proceedings of the 2nd ACM conference on Electronic commerce*, pages 66–76. ACM, 2000.