

The Vickrey-Clarke-Groves Mechanism

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We introduce the Vickrey-Clarke-Groves Mechanism mechanism, a direct mechanism for multi-parameter settings, and argue that it is economically efficient and DSIC, assuming independent private values.

1 Combinatorial Auctions

In single-parameter environments, there is usually a single good, but there can be more than one good (e.g., sponsored search). Either way, the bidders are always characterized by but one parameter. In more general environments, there may be combinatorially many outcomes (e.g., assuming multiple goods, all assignments of bundles of goods to bidders), so the bidders' valuations cannot easily be characterized by a single parameter. Auctions designed for *multi-parameter* environments are called **combinatorial auctions**.

More formally, when bidders' preferences are combinatorial, we assume they can be described by **valuations**, which are functions from outcomes to values: i.e., $v_i : \Omega \rightarrow \mathbb{R}$. Therefore, in direct mechanisms, the focus of the present lecture, bids take the form of *functions* from outcomes to values: i.e., $b_i : \Omega \rightarrow \mathbb{R}$.

As usual, we are driven by three auction design goals:

1. incentives: we seek auctions in which reporting bids truthfully is an equilibrium, preferably in dominant strategies (i.e., DSIC)
2. economic efficiency (e.g., welfare maximization)
3. computational efficiency

It should be immediately apparent that designing such an auction in a computationally efficient manner is a non-trivial endeavor. Just describing a single bidder's valuation—and hence, communicating bids to the auctioneer—can take exponential time and space. We proceed nonetheless, temporarily abandoning the third design goal.

We make the following two assumptions about bidders' valuations throughout this lecture:

1. Valuations are **normalized**, so that each bidder's value for an outcome in which she is assigned the empty bundle is zero.
2. Valuations are **monotone**, so that each bidder's values are weakly increasing in the size of her assigned bundle.

The monotonicity assumption is also called **free disposal**, as it costs the bidders nothing to accrue additional goods.

Note that together these assumptions imply that no bidder's value for any bundle is negative: i.e., for all $i \in [n]$ and for all $\omega \in \Omega$, $v_i(\omega) \geq 0$ (since the value of the empty bundle is zero, and accruing additional goods can only increase this value).

2 The Vickrey-Clarke-Groves Mechanism

Recall that an auction outcome comprises an allocation and a payment rule. As utilities are quasilinear, it suffices to consider auction outcomes as allocations only. In other words, we assume Ω is the space of possible (feasible) allocations of goods to bidders.

The standard procedure for designing a DSIC auction is as follows:

1. Find an economically efficient (i.e., welfare-maximizing) allocation: i.e., given bids \mathbf{b} , find an allocation ω^* s.t.

$$\omega^* \in \arg \max_{\omega \in \Omega} \sum_{i \in [n]} b_i(\omega).$$

This problem is called the **winner determination problem**.

2. Charge each winner an appropriate payment, so as to ensure the DSIC property holds.

The complexity of the winner determination problem can make it very difficult, if not impossible, to deploy auctions designed via this procedure in practice. We return to this issue later. For the moment, we seek to explore the philosophical question, is there a payment formula that ensures the DSIC property? The answer is the following:

$$p_i(\omega^*) = h_i(\mathbf{b}_{-i}) - \sum_{j \neq i \in [n]} b_j(\omega^*)$$

where, as above, ω^* denotes an efficient allocation, given bids \mathbf{b} . A mechanism that uses this payment formula is called a **Groves** mechanism. Before we unpack this formula, let's prove that the Groves mechanism is in fact DSIC.

Our proof and the discussion that follows rely on the following notation: Given a bidder i and a bid profile \mathbf{b}_{-i} , we write $f(b_i) \doteq f(b_i, \mathbf{b}_{-i})$ to denote an efficient allocation, assuming i reports b_i : i.e.,

$$f(b_i) \in \arg \max_{\omega \in \Omega} \sum_{i \in [n]} b_i(\omega).$$

Bidder i 's utility at such an efficient allocation is then

$$u_i(b_i, \mathbf{b}_{-i}) = v_i(f(b_i)) - \left(h_i(\mathbf{b}_{-i}) - \sum_{j \neq i \in [n]} b_j(f(b_i)) \right) \quad (1)$$

Theorem 2.1. *The Groves mechanism is DSIC.*

Proof. Bidder i 's utility is given by

$$u_i(b_i, \mathbf{b}_{-i}) = v_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)) - h_i(\mathbf{b}_{-i}). \quad (2)$$

Note that $h_i(\mathbf{b}_{-i})$ is a constant, independent of bidder i 's report.

Hence, if bidder i had the power to choose the allocation, she would choose one that maximizes

$$v_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)).$$

Although she does not have this power, she can maximize this value by reporting truthfully, because the VCG mechanism, which seeks to optimize

$$b_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)),$$

would thus select an optimizing allocation on her behalf. \square

Beyond incentive compatibility, the other important incentive property is individual rationality. That is, bidders' participation in the mechanism should be voluntary. Another way to state this objective is that bidders' utilities should always be non-negative, regardless of the outcome. Recall that bidder i 's utility is

$$u_i(b_i, \mathbf{b}_{-i}) = v_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)) - h_i(\mathbf{b}_{-i}).$$

To ensure that this quantity is non-negative, we require

$$v_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)) \geq h_i(\mathbf{b}_{-i}). \quad (3)$$

We can satisfy the desired inequality by letting $h_i(\mathbf{b}_{-i})$ be the sum total of all bidders' bids, *except bidder i 's*, at an efficient allocation: i.e.,

$$h_i(\mathbf{b}_{-i}) = \max_{\omega \in \Omega} \sum_{j \neq i \in [n]} b_j(\omega),$$

because $f(b_i)$ is an efficient allocation, and all bidders' values are non-negative at all outcomes, so adding bidder i into the mix (on the LHS of Equation 3) cannot decrease welfare.

This choice of $h_i(\mathbf{b}_{-i})$ is called the **Clarke pivot rule**. Auctions, or more generally mechanisms, that use the aforementioned design process together with the Clarke pivot rule in the payment formula are called **Vickrey-Clarke-Grove (VCG) mechanisms**.¹

With the Clarke pivot rule and non-negative valuations, the VCG mechanism is individually rational (IR): i.e., participation is voluntary. Moreover, the VCG mechanism never pays bidders to participate (i.e., all payments are non-negative).

¹ Vickrey auctions are a special case of VCG mechanisms, the latter of which apply beyond auctions: e.g., in the realm of public goods provisioning.

In spite of its desirable properties (e.g., DSIC and IR), the VCG mechanism still exhibits some bizarre behavior. You will explore the following anomalies in this week's homework exercises:

1. The VCG mechanism may allocate goods to bidders with strictly positive valuations, and still generate zero revenue.
2. The VCG mechanism may generate less revenue for the auctioneer when an additional bidder participates.
3. The VCG mechanism may generate more utility for bidders who collude to submit untruthful bids. Indeed, bidders can collude with themselves by submitting what are called **false-name** bids!

3 Interpreting the VCG Payment Formula

We now describe two ways to interpret the VCG payment formula.

When bidder i is present, the other bidders, namely $[n] \setminus \{i\}$, may not achieve as much welfare (collectively) as they do when i is not present. With bidder i present, the welfare generated is

$$\omega^* \in \arg \max_{\omega \in \Omega} \sum_{j \in [n]} v_j(\omega),$$

whereas

$$\max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega)$$

is the welfare that would be generated were bidder i not present. The net difference in welfare for the set of bidders $[n] \setminus \{i\}$ is

$$\max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega) - \sum_{j \neq i \in [n]} v_j(\omega^*).$$

This quantity is exactly the payment the VCG mechanism charges bidder i . Thus, we can view bidder i 's payment as the **externality** she imposes on all the other bidders, collectively.

Example 3.1. Assume a single-good auction, in which bidder i has the highest value for the good. At the efficient allocation ω^* , $\sum_{j \neq i \in [n]} v_j(\omega^*) = 0$, so the VCG mechanism charges the winner $\max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega)$, which (not coincidentally!) is the second-highest bid. Thus, the second-price, sealed-bid auction (a.k.a., the Vickrey auction) is the VCG mechanism assuming only one good.

Another way of interpreting the payment formula is in terms of rebates. Expand the payment formula as follows:

$$p_i(\omega^*) = \max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega) - \sum_{j \neq i \in [n]} v_j(\omega^*)$$

$$\begin{aligned}
&= v_i(\omega^*) + \max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega) - \sum_{j \neq i \in [n]} v_j(\omega^*) - v_i(\omega^*) \\
&= v_i(\omega^*) - \left[\sum_{j \in [n]} v_j(\omega^*) - \max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega) \right].
\end{aligned}$$

We see that bidder i 's payments are precisely her value less a non-negative quantity. This quantity can be understood as the amount of additional welfare that can be attributed to i 's presence: i.e., the value of the efficient allocation with i less the value of the efficient allocation without i .

Example 3.2. In a Vickrey auction for a single good, suppose, without loss of generality, bidder 1 wins (ω^*), and bidder 2's bid is the second highest. Bidder 1 pays $p_1(\omega^*) = v_1 - (v_1 - v_2) = v_2$.

Here is present an example of how the VCG mechanism operates with three bidders and two goods.

Example 3.3. Suppose there are three bidders and two goods, A and B , with valuations as described in Table 1.

Bundle	v_1	v_2	v_3
\emptyset	0	0	0
$\{A\}$	2	1	1
$\{B\}$	2	1	1
$\{A, B\}$	2	1	4

Table 1: Bidder valuations for winning subsets of goods in $G = \{A, B\}$.

The efficient allocation ω^* gives bundle $\{A, B\}$ to bidder 3, and results in total welfare of 4.

- Without bidder 1, the maximum possible welfare would be 4. Bidder 1's contribution to welfare is zero. Payments for bidder 1 are

$$p_1(\omega^*) = 4 - 4 = 0 - (4 - 4) = 0.$$

- Without bidder 2, the maximum possible welfare would be 4. Bidder 2's contribution to welfare is zero. Payments for bidder 2 are

$$p_2(\omega^*) = 4 - 4 = 0 - (4 - 4) = 0.$$

- Without bidder 3, the maximum possible welfare would be 3. (We would allocate A to bidder 1 and B to bidder 2.) Payments for bidder 3 are

$$p_3(\omega^*) = 3 - 0 = 4 - (4 - 3) = 3.$$