Myerson's Optimal Auction Design

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We prove Myerson's seminal result,[1] that total expected revenue equals total expected virtual value in a DSIC, IR single-parameter auction.

[1] Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981

1 *Mathematical Aside*

Integration is usually performed with respect to the x axis:

$$\int_a^b f(x)\,\mathrm{d}x$$

But sometimes (e.g., in the case of Myerson's formula), it can be convenient to integrate with respect to the y axis instead:[2]

[2] assuming f^{-1} exists

$$\int_{f(a)}^{f(b)} x\,\mathrm{d}y = \int_{f(a)}^{f(b)} f^{-1}(y)\,\mathrm{d}y$$

When integrating along the x-axis, we "sum" all the vertical rectangles from a to b with width x and height $f(x)$. Similarly, when integrating along the y-axis, we "sum" all the horizontal rectangles from $f(a)$ to $f(b)$ with height y and width $x = f^{-1}(y)$.

We can likewise express this latter summation as integrating along the x-axis, i.e., as the "sum" of vertical rectangles from a to b with width x and height $\mathrm{d}y = \mathrm{d}f(x)$:

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,\mathrm{d}y = \int_a^b x\,\mathrm{d}f(x)$$

2 *Myerson's Lemma: Recap*

Recall Myerson's payment characterization formula, assuming $p_i(\underline{v}_i, \mathbf{v}_{-i}) - \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) = c$, namely:

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i})\,\mathrm{d}z + c. \tag{1}$$

In our derivation of Myerson's optimal auction, we rely on an equivalent payment characterization formula, in which we integrate the allocation function $x_i(z, \mathbf{v}_{-i})$ along the y-axis, as follows:

$$p_i(v_i, \mathbf{v}_{-i}) = \int_{\underline{v}_i}^{v_i} z\,\mathrm{d}x_i(z, \mathbf{v}_{-i}) + c. \tag{2}$$

3 Optimal Auction Design

In this lecture, using Myerson's lemma as a starting point, we will prove Myerson's theorem, which dictates the design of an optimal (i.e., revenue-maximizing) auction. This theorem relates payments (i.e., revenue) to **virtual welfare**, which, as we will see, is defined in terms of each bidder's **virtual value** function φ_i.

Like Myerson's lemma, the theorem concerns a single-parameter auction. Furthermore, each bidder i's values v_i are drawn from a continuous distribution F_i, with support $T_i = [\underline{v}_i, \overline{v}_i]$, for some lowest and highest types $\underline{v}_i, \overline{v}_i \in \mathbb{R}_+$.

Theorem 3.1 (Myerson)**.** *In a DSIC, IR*[3] *single-parameter auction, the revenue the auctioneer can expect to collect from bidder $i \in [n]$ is equal to bidder i's expected virtual welfare: i.e.,*

$$\mathop{\mathbb{E}}_{v_i \sim F_i}\left[p_i(v_i, \mathbf{v}_{-i})\right] = \mathop{\mathbb{E}}_{v_i \sim F_i}\left[\varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})\right], \tag{3}$$

assuming $v_i \sim F_i$; $f_i(v_i) > 0$, for all $v_i \in T_i$; and $T_i = \left[\underline{v}_i, \overline{v}_i\right]$, for $-\infty < \underline{v}_i < \overline{v}_i < \infty$.

[3] We assume the lowest type is allocated nothing and pays nothing, so that $\underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i}) = 0$.

Proof. For a fixed $\mathbf{v}_{-i} \in T_{-i}$, bidder i's contribution to total expected revenue is as follows:

$$\begin{aligned}
\mathop{\mathbb{E}}_{v \sim F_i}\left[p_i(v, \mathbf{v}_{-i})\right] &= \int_{\underline{v}_i}^{\overline{v}_i} p_i(v, \mathbf{v}_{-i}) f_i(v)\, \mathrm{d}v \\
&= \int_{\underline{v}_i}^{\overline{v}_i} \left[\int_{\underline{v}_i}^{v} z\, \mathrm{d}x_i(z, \mathbf{v}_{-i})\right] f_i(v)\, \mathrm{d}v \\
&= \int_{\underline{v}_i}^{\overline{v}_i} \left[\int_{\underline{v}_i}^{v} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z}\right) \mathrm{d}z\right] f_i(v)\, \mathrm{d}v \\
&= \int_{\underline{v}_i}^{\overline{v}_i} \int_{\underline{v}_i}^{v} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z}\right) f_i(v)\, \mathrm{d}z\, \mathrm{d}v.
\end{aligned}$$

Now, note that the allocation function is non-negative, that $f_i(v)$ is positive by assumption, and that the type space is bounded s.t. the bid z is also non-negative. Consequently, $z\left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z}\right) f_i(v)$ is a non-negative function. By Tonelli's theorem the order of the integration can therefore be reversed.

To reverse the order of integration we must also change the limits of integration. Notice that v ranges from $\underline{v}_i$ to $\overline{v}_i$, and that, for any fixed $v = V$, z ranges from $\underline{v}_i$ to V. If we instead let z range from $\underline{v}_i$ to $\overline{v}_i$, then for any fixed $z = Z$, v ranges from Z to $\overline{v}_i$. We attempt to depict this argument (Tonelli's theorem) in Figure 1.

Using our knowledge of the region of integration, we can switch the order of integration as follows:

$$\int_{\underline{v}_i}^{\overline{v}_i} \int_{\underline{v}_i}^{v} z \left(\frac{\mathrm{d}x_i(z, \mathbf{v}_{-i})}{\mathrm{d}z}\right) f_i(v)\, \mathrm{d}z\, \mathrm{d}v$$

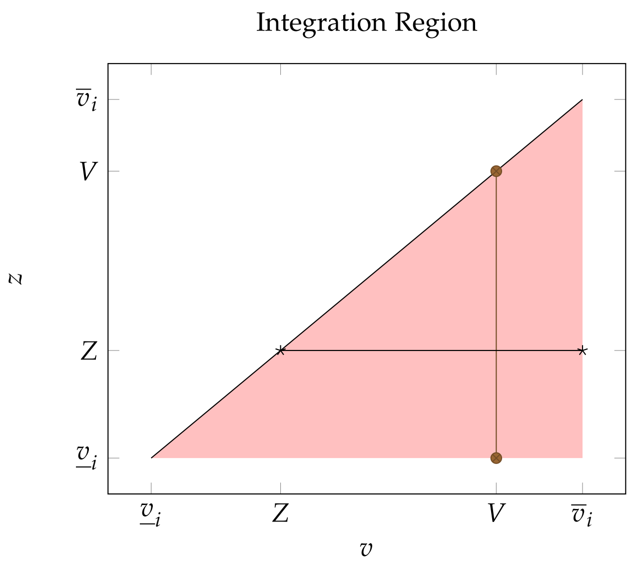

Figure 1: Tonelli's theorem: The shaded area represents the values of v and z used to evaluate bidder i's contribution to expected revenue.

$$
\begin{aligned}
&= \int_{\underline{v}_i}^{\overline{v}_i} \int_{z}^{\overline{v}_i} z \left(\frac{\mathrm{d} x_i(z, \mathbf{v}_{-i})}{\mathrm{d} z} \right) f_i(v) \, \mathrm{d} v \, \mathrm{d} z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} z \left(\frac{\mathrm{d} x_i(z, \mathbf{v}_{-i})}{\mathrm{d} z} \right) \int_{z}^{\overline{v}_i} f_i(v) \, \mathrm{d} v \, \mathrm{d} z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} z \left(\frac{\mathrm{d} x_i(z, \mathbf{v}_{-i})}{\mathrm{d} z} \right) F_i(v) \Big|_{z}^{\overline{v}_i} \mathrm{d} z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} z \left(\frac{\mathrm{d} x_i(z, \mathbf{v}_{-i})}{\mathrm{d} z} \right) (1 - F_i(z)) \, \mathrm{d} z.
\end{aligned}
$$

Next, we use integration by parts:

$$
\int_a^b u \, \mathrm{d} v = uv|_a^b - \int_a^b v \, \mathrm{d} u,
$$

where we let

$$
\begin{aligned}
u &= z \left[1 - F_i(z) \right] & \mathrm{d} u &= \left[-z f_i(z) + 1 - F_i(z) \right] \, \mathrm{d} z \\
\mathrm{d} v &= \left(\frac{\mathrm{d} x_i(z, \mathbf{v}_{-i})}{\mathrm{d} z} \right) \mathrm{d} z & v &= x_i(z, \mathbf{v}_{-i}),
\end{aligned}
$$

to get:

$$
\begin{aligned}
&\int_{\underline{v}_i}^{\overline{v}_i} z \left(\frac{\mathrm{d} x_i(z, \mathbf{v}_{-i})}{\mathrm{d} z} \right) (1 - F_i(z)) \, \mathrm{d} z \\
&= \underbrace{z \, (1 - F_i(z))}_{u} \underbrace{x_i(z, \mathbf{v}_{-i})}_{v} \Big|_{\underline{v}_i}^{\overline{v}_i} - \int_{\underline{v}_i}^{\overline{v}_i} \underbrace{x_i(z, \mathbf{v}_{-i})}_{v} \underbrace{(-z f_i(z) + 1 - F_i(z))}_{du} \, \mathrm{d} z \\
&= (0 - 0) + \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i}) \, (z f_i(z) - (1 - F_i(z))) \, \mathrm{d} z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i}) \, (z f_i(z) - (1 - F_i(z))) \, \mathrm{d} z.
\end{aligned}
$$

What we have at this point is not quite an expectation. But, since $f_i(z) > 0$ for all $z \in T_i$, multiplying by $f_i(z)/f_i(z)$ does no harm:

$$1 - F_i(z) = \left(\frac{1 - F_i(z)}{f_i(z)}\right) f_i(z).$$

After this manuever, our expression simplifies as follows:

$$\begin{aligned}
&\int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i})\,(z f_i(z) - (1 - F_i(z)))\,\mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i}) \left(z f_i(z) - \left(\frac{1 - F_i(z)}{f_i(z)}\right) f_i(z)\right) \mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i}) \left(z - \frac{1 - F_i(z)}{f_i(z)}\right) f_i(z)\,\mathrm{d}z \\
&= \int_{\underline{v}_i}^{\overline{v}_i} x_i(z, \mathbf{v}_{-i}) \varphi_i(z) f_i(z)\,\mathrm{d}z \\
&= \mathop{\mathbb{E}}_{z \sim F_i} \left[\varphi_i(z) x_i(z, \mathbf{v}_{-i})\right],
\end{aligned}$$

where $\varphi_i(z) = z - \frac{1 - F_i(z)}{f_i(z)}$. We call this quantity **virtual value**. Correspondingly, we call the quantity $\varphi_i(v) x_i(v, \mathbf{v}_{-i})$ **virtual welfare**, since it is allocation times virtual value, instead of allocation times value.

Finally, renaming the bound variable z as the usual v yields:

$$\mathop{\mathbb{E}}_{v \sim F_i} \left[p_i(v, \mathbf{v}_{-i})\right] = \mathop{\mathbb{E}}_{v \sim F_i} \left[\varphi_i(v) x_i(v, \mathbf{v}_{-i})\right]$$

□

Remark 3.2. More generally, if $p_i(\underline{v}_i, \mathbf{v}_{-i}) - \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) = c \neq 0$, then

$$\mathop{\mathbb{E}}_{v_i \sim F_i} \left[p_i(v_i, \mathbf{v}_{-i})\right] = \mathop{\mathbb{E}}_{v_i \sim F_i} \left[\varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})\right] + c. \tag{4}$$

Now that we can relate a single bidder i's contribution to total expected revenue to its contribution to total expected virtual welfare, it is straightforward to show that *total* expected revenue is equal to *total* expected virtual welfare.

Corollary 3.3 (Myerson)**.** *In a DSIC, IR single-parameter auction, total expected revenue is equal to total expected virtual welfare: i.e.,*

$$\mathop{\mathbb{E}}_{\mathbf{v} \sim F} \left[\sum_{i \in [n]} p_i(v_i, \mathbf{v}_{-i})\right] = \mathop{\mathbb{E}}_{\mathbf{v} \sim F} \left[\sum_{i \in [n]} \varphi_i(v_i) x_i(v_i, \mathbf{v}_{-i})\right], \tag{5}$$

assuming, for all players $i \in [n]$, $v_i \sim F_i$; $f_i(v_i) > 0$, for all $v_i \in T_i$; and $T_i = [\underline{v}_i, \overline{v}_i]$, for $-\infty < \underline{v}_i < \overline{v}_i < \infty$.

Proof. Take expectations with respect to $\mathbf{v}_{-i}$ to get

$$\mathop{\mathbb{E}}_{\mathbf{v}_{-i} \sim F_{-i}} \left[\mathop{\mathbb{E}}_{v \sim F_i} \left[p_i(v, \mathbf{v}_{-i})\right]\right] = \mathop{\mathbb{E}}_{\mathbf{v}_{-i} \sim F_{-i}} \left[\mathop{\mathbb{E}}_{v \sim F_i} \left[\varphi_i(v) x_i(v, \mathbf{v}_{-i})\right]\right],$$

or equivalently,

$$\mathop{\mathbb{E}}_{\mathbf{v}\sim F}[p_i(\mathbf{v})] = \mathop{\mathbb{E}}_{\mathbf{v}\sim F}[\varphi_i(v_i)x_i(\mathbf{v})].$$

Then, by linearity of expectations, we complete the proof:

$$\mathop{\mathbb{E}}_{\mathbf{v}\sim F}\left[\sum_{i\in[n]} p_i(\mathbf{v})\right] = \sum_{i\in[n]} \mathop{\mathbb{E}}_{\mathbf{v}\sim F}[p_i(\mathbf{v})] = \sum_{i\in[n]} \mathop{\mathbb{E}}_{\mathbf{v}\sim F}[\varphi_i(v_i)x_i(\mathbf{v})] = \mathop{\mathbb{E}}_{\mathbf{v}\sim F} \sum_{i\in[n]} [\varphi_i(v_i)x_i(\mathbf{v})].$$

□

Myerson's theorem tells us that we can maximize revenue by maximizing virtual welfare. In other words, only bidders with the highest virtual values should be allocated. Moreover, bidders should only be allocated if their virtual values are non-negative; otherwise, virtual welfare would not be maximized.

Myerson's lemma further tells us that incentive compatibility mandates a monotone allocation rule: i.e., as values (weakly) increase, so, too, should the allocation function. But as we are to allocate by virtual values not values themselves in the optimal auction, ensuring (weak) monotonicity requires a further assumption, namely that virtual values are (weakly) increasing in values.

A distribution is called **regular** if the corresponding virtual value function is weakly increasing in values. The regularity assumption ensures that allocating in order of virtual values is monotone, from which it follows that the optimal auction is incentive compatible.

References

[1] Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.