The Envelope Theorem

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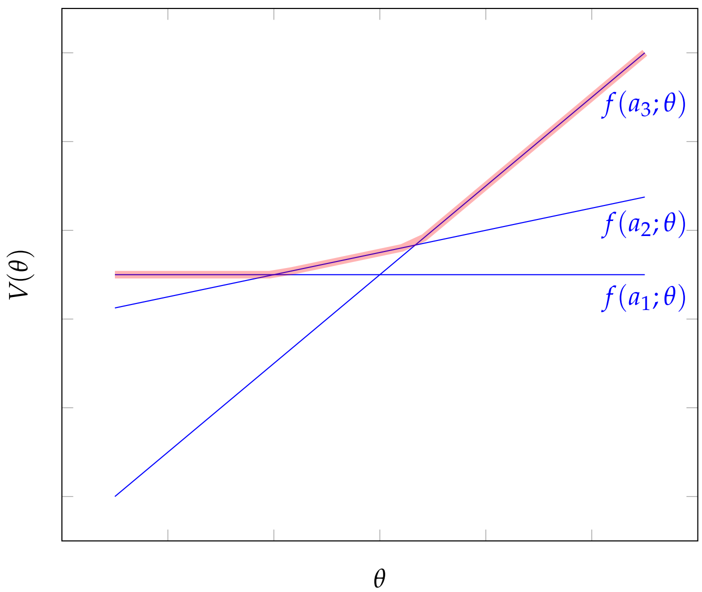
We prove the celebrated envelope theorem. Then, by way of this theorem, we derive the symmetric equilibrium in first-price auctions and Myerson's payment characterization for DSIC auctions. We follow the presentation in Quint.[1]

[1] Dan Quint. Some beautiful theorems with beautiful proofs. University of Wisconsin–Madison, 2014

1 Envelope Theorem

We prove the envelope theorem in its simplest form. In more interesting/complicated versions, the constraint set depends on the parameter θ.



Consider an optimization problem

$$V(\theta) = \max_{a \in A} f(a;\theta) \ ,$$

whose objective is parameterized by some $\theta \in \Theta$. Implicit in this problem is the search for an optimal strategy $s^* : \Theta \to A$ from the parameter space Θ to the action space A.

Theorem 1.1. *Let $A^*(\theta) = \arg\max_{a \in A} f(a;\theta)$, and assume $A^*(\theta)$ is nonempty, with $s^*(\theta)$ an element of $A^*(\theta)$, so that $V(\theta) = f(s^*(\theta);\theta)$. If $V(\theta)$ and $f(a;\theta)$, for all $a \in A$, are differentiable at $\theta \in \Theta$, then*

$$V'(\theta) = \frac{\mathrm{d} f(s^*(\theta);\theta)}{\mathrm{d}\theta} \ .$$

Proof. By definition, if V is differentiable at θ, then

$$V'(\theta) = \lim_{\epsilon \to 0} \frac{V(\theta + \epsilon) - V(\theta)}{\epsilon} = \lim_{\epsilon \to 0} \frac{V(\theta) - V(\theta - \epsilon)}{\epsilon} \ .$$

Since $V(\theta+\epsilon) = \max_{a\in A} f(a;\theta+\epsilon) \geq f(s^*(\theta);\theta+\epsilon)$, it follows that

$$V'(\theta) = \lim_{\epsilon\to 0} \frac{V(\theta+\epsilon)-V(\theta)}{\epsilon} \geq \lim_{\epsilon\to 0} \frac{f(s^*(\theta);\theta+\epsilon) - f(s^*(\theta);\theta)}{\epsilon} = \frac{\mathrm{d}f(s^*(\theta);\theta)}{\mathrm{d}\theta} \ .$$

Similarly, since $V(\theta-\epsilon) = \max_{a\in A} f(a;\theta-\epsilon) \geq f(s^*(\theta);\theta-\epsilon)$, it follows that

$$V'(\theta) = \lim_{\epsilon\to 0} \frac{V(\theta)-V(\theta-\epsilon)}{\epsilon} \leq \lim_{\epsilon\to 0} \frac{f(s^*(\theta);\theta) - f(s^*(\theta);\theta-\epsilon)}{\epsilon} = \frac{\mathrm{d}f(s^*(\theta);\theta)}{\mathrm{d}\theta} \ .$$

The result now follows, as

$$\frac{\mathrm{d}f(s^*(\theta);\theta)}{\mathrm{d}\theta} \leq V'(\theta) \leq \frac{\mathrm{d}f(s^*(\theta);\theta)}{\mathrm{d}\theta} \ .$$

□

Example 1.2. The following functions are shown in the figure above:

$$f(a_1;\theta) = 1 \quad (1)$$

$$f(a_2;\theta) = \theta/4 + 3/2 \quad (2)$$

$$f(a_3;\theta) = \theta + 1 \quad (3)$$

Observe that $f(a;\theta)$ is differentiable, for all $a \in A$, and at all $\theta \in \Theta$. Although $V(\theta)$ is *not* differentiable at $\{-2, 2/3\}$, the envelope theorem gives:

$$V'(\theta) = \begin{cases} \frac{\mathrm{d}f(a_1(\theta);\theta)}{\mathrm{d}\theta} & \theta < -2 \\ \frac{\mathrm{d}f(a_2(\theta);\theta)}{\mathrm{d}\theta} & -2 < \theta < 2/3 \\ \frac{\mathrm{d}f(a_3(\theta);\theta)}{\mathrm{d}\theta} & 2/3 < \theta \end{cases}$$

$$= \begin{cases} 0 & \theta < -2 \\ 1/4 & -2 < \theta < 2/3 \\ 1 & 2/3 < \theta \end{cases}$$

Corollary 1.3. *Assume the conditions of Theorem 1.1. If, additionally, $f(a;z)$ is a continuous, real-valued function on the closed interval $[\theta,\tau]$, then*

$$V(\tau) - V(\theta) = \int_\theta^\tau \frac{\mathrm{d}f(a^*(z);z)}{\mathrm{d}z}\,\mathrm{d}z$$

Proof. Combining the fundamental theorem of calculus and the envelope theorem yields:

$$\begin{aligned} V(\tau) - V(\theta) &= \int_\theta^\tau V'(z)\,\mathrm{d}z \\ &= \int_\theta^\tau \frac{\mathrm{d}f(a^*(z);z)}{\mathrm{d}z}\,\mathrm{d}z \end{aligned}$$

□

2 *Key Observation*

Recall that an auction is defined by two rules, an allocation rule and a payment rule. We consider a single-parameter auction for one good in which each bidder's valuation/value for the good is described by one number, $v_i \in \mathbb{R}_+$. Fixing all other agents' bids $\mathbf{b}_{-i}$, bidder i's payment is $p_i(b_i)$, so that her quasilinear utility if she is allocated the good is $v_i - p_i(b_i)$. Her probability of allocation is $x_i(b_i) \in [0, 1]$, so that her expected utility is $x_i(b_i)(v_i - p_i)(b_i)$. In this setting, the envelope theorem yields an interesting insight about the optimal expected utility function, namely that its derivative is the allocation function.

Lemma 2.1. *Given a bidder i with value v_i and quasilinear utility function $u_i(b_i) = x_i(b_i)(v_i - p_i(b_i))$, so that her* optimal *utility function is given by*

$$U^*(v_i) = \max_{b_i \in B_i} x_i(b_i)(v_i - p_i(b_i)) \ .$$

The derivative of her optimal utility function with respect to her value is her allocation function: i.e.,

$$\frac{\mathrm{d}U^*(v_i)}{\mathrm{d}v_i} = x_i(s_i^*(v_i)) \ ,$$

where $s_i^*(v_i)$ denotes a utility-maximizing bid.

Proof. First,

$$U_i'(v_i) = \frac{\mathrm{d}}{\mathrm{d}v_i} \max_{b_i \in B_i} x_i(b_i)(v_i - p_i(b_i)) \tag{4}$$

$$= \frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i))(v_i - p_i(s_i^*(v_i))) \tag{5}$$

Equation 4 follows from the definition of optimal expected utility. Equation 5 follows from the envelope theorem. Next, via the sum rule of calculus:

$$\frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i))(v_i - p_i(s_i^*(v_i))) \tag{6}$$

$$= \frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i))v_i - \frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i))p_i(s_i^*(v_i)) \ . \tag{7}$$

Applying the product rule to the first term of Equation 7 yields:

$$\frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i))v_i = \frac{\mathrm{d}x_i(s_i^*(v_i))}{\mathrm{d}v_i} v_i + x_i(s_i^*(v_i)) \frac{\mathrm{d}v_i}{\mathrm{d}v_i} \tag{8}$$

$$= \frac{\mathrm{d}x_i(s_i^*(v_i))}{\mathrm{d}v_i} v_i + x_i(s_i^*(v_i)) \ . \tag{9}$$

Applying it again to the second term yields:

$$\frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i))p_i(s_i^*(v_i)) \tag{10}$$

$$= \frac{\mathrm{d}x_i(s_i^*(v_i))}{\mathrm{d}v_i} p_i(s_i^*(v_i)) + x_i(s_i^*(v_i)) \frac{\mathrm{d}p_i(s_i^*(v_i))}{\mathrm{d}v_i} \quad . \tag{11}$$

Therefore,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i)) v_i - \frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i)) p_i(s_i^*(v_i)) \\ =& \frac{\mathrm{d}x_i(s_i^*(v_i))}{\mathrm{d}v_i} v_i + x_i(s_i^*(v_i)) \\ +& \frac{\mathrm{d}x_i(s_i^*(v_i))}{\mathrm{d}v_i} p_i(s_i^*(v_i)) + x_i(s_i^*(v_i)) \frac{\mathrm{d}p_i(s_i^*(v_i))}{\mathrm{d}v_i} \quad . \end{aligned} \tag{12}$$

Further simplification now requires the chain rule.

Recall the chain rule: if $y = f(u)$ and $u = g(v)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \frac{\mathrm{d}y}{\mathrm{d}u} \frac{\mathrm{d}u}{\mathrm{d}v}$$

Letting $f = x_i(u)$ and $g = s_i^*(v_i)$, so that $f = x_i(s_i^*(v_i))$, yields:

$$\frac{\mathrm{d}x_i(s_i^*(v_i))}{\mathrm{d}v_i} = \frac{\mathrm{d}x_i(s_i^*(v_i))}{\mathrm{d}s_i^*(v_i)} \frac{\mathrm{d}s_i^*(v_i)}{\mathrm{d}v_i} \tag{13}$$

But this second term is necessarily zero, since s_i^* is utility maximizing. In other words, $\frac{\mathrm{d}x_i(s_i^*(v_i))}{\mathrm{d}v_i} = 0$. Similarly, $\frac{\mathrm{d}p_i(s_i^*(v_i))}{\mathrm{d}v_i} = 0$.

Therefore, Equation 12 simplifies as follows:

$$\frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i)) v_i - \frac{\mathrm{d}}{\mathrm{d}v_i} x_i(s_i^*(v_i)) p_i(s_i^*(v_i)) = x_i(s_i^*(v_i)) \quad , \tag{14}$$

and the theorem is proved. □

3 *Myerson's Payment Formula via the Envelope Theorem*

Finally, again using the envelope theorem, we prove (one direction of) Myerson's payment characterization theorem—that the DSIC assumption implies Myerson's payment formula.

Proof. Reverting back to our original notation, we denote bidder i's optimal utility (or value function) by V_i. By Lemma 2.1, $V_i'(v_i) = x_i(s_i^*(v_i))$. Moreover, by the DSIC assumption, bidder i's expected utility is maximized at v_i: i.e., $s_i^*(v_i) = v_i$. Therefore, $V_i'(v_i) = x_i(v_i)$.

More specifically, $V_i'(v_i, \mathbf{v}_{-i}) = x_i(v_i, \mathbf{v}_{-i})$. But then, by Corollary 1.3 (which invokes the fundamental theorem of calculus),

$$V_i(v_i, \mathbf{v}_{-i}) - V_i(\underline{v}_i, \mathbf{v}_{-i}) = \int_{\underline{v}_i}^{v_i} V_i'(z, \mathbf{v}_{-i}) \, \mathrm{d}z = \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i}) \, \mathrm{d}z \quad .$$

Next, letting $p_i(v_i, \mathbf{v}_{-i})$ denote bidder i's *expected* payment, we can also express bidder i's expected utility $V_i(v_i, \mathbf{v}_{-i})$ as $v_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i})$.

It now follows that

$$v_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i}) - \left(\underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) - p_i(\underline{v}_i, \mathbf{v}_{-i})\right) = \int_{\underline{v}_i}^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z \enspace .$$

In other words,

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \int_{\underline{v}_i}^{v_i} x_i(v_i, \mathbf{v}_{-i})\, \mathrm{d}z + p_i(\underline{v}_i, \mathbf{v}_{-i}) - \underline{v}_i x_i(\underline{v}_i, \mathbf{v}_{-i}) \enspace .$$

□

4 *Equilibrium Derivations via the Envelope Theorem*

Next, let's use the envelope theorem to analyze a symmetric first-price auction, meaning one in which the bidders' values are drawn i.i.d. from a bounded distribution F on $[\underline{v}, \overline{v}]$, for some $\underline{v} \leq \overline{v} \in \mathbb{R}$.

Assume a symmetric equilibrium $s^*(v)$ that is non-decreasing in v, so that a bidder with the highest value wins. At such an equilibrium, if the allocation probability is $x(v)$ and the winner pays her bid $s^*(v)$, each bidder's expected utility is given by $U^*(v) = x(v)(v - s^*(v))$. The probability that a bidder i with value v wins is the probability that $v \geq v_j$, for all $j \neq i$: i.e., $F^{n-1}(v)$. Each bidder's expected utility is thus:

$$U^*(v) = F^{n-1}(v)(v - s^*(v)) \enspace . \tag{15}$$

By Lemma 2.1, $\frac{\mathrm{d}U^*(v)}{\mathrm{d}v} = x(s^*(v))$. Moreover, by the monotonicity assumption (i.e., $s^*(v)$ that is non-decreasing in v), $x(s^*(v)) = F^{n-1}(v)$. Therefore, by the fundamental theorem of calculus,

$$U^*(v) = \int_{\underline{v}}^{v} F^{n-1}(t)\, \mathrm{d}t \enspace , \tag{16}$$

as $U^*(\underline{v}) = 0$. Setting these two expressions for $U^*(v)$ (Equations 15 and 16) equal to one another yields

$$F^{n-1}(v)(v - s^*(v)) = \int_{\underline{v}}^{v} F^{n-1}(t)\, \mathrm{d}t \enspace , \tag{17}$$

from which it follows that

$$s^*(v) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}(t)\, \mathrm{d}t}{F^{n-1}(v)} \enspace . \tag{18}$$

Finally, since[2]

$$vF^{n-1}(v) - \int_{\underline{v}}^{v} F^{n-1}(t)\, \mathrm{d}t = \int_{\underline{v}}^{v} t\, \mathrm{d}F^{n-1} \enspace , \tag{19}$$

it follows that

$$s^*(v) = \frac{\int_{\underline{v}}^{v} t\, \mathrm{d}F^{n-1}}{F^{n-1}(v)} \enspace . \tag{20}$$

[2] See Math'l Aside at the start of Lecture 5 on Myerson's optimal auction design.

In other words, at equilibrium in a symmetric first-price auction, bidders shade their bids in such a way that the result is the expected bid of the bidder with the second-highest value, conditioned on their value being highest.

Remark 4.1. This derivation establishes *necessary* conditions for s^* to be a symmetric equilibrium of a symmetric first-price auction: i.e., if s^* is such an equilibrium, then it must take the form of Equation 18 (equivalently, Equation 20). Last week, we proved sufficiency in the special case where $F = U[0,1]$, namely that $s^*(v) = \left(\frac{n-1}{n}\right) v$ is a best response to itself.

References

[1] Dan Quint. Some beautiful theorems with beautiful proofs. University of Wisconsin–Madison, 2014.