Regular and MHR Distributions

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We define the hazard rate function, and then regular and monotone hazard rate (MHR) distributions, and we contrast MHR distributions with regular distributions.

1 *An Aside: Survival Distributions*

Assume T is a continuous random variable representing whether an event (e.g., marriage, parenthood, migration, death, etc.) has occurred by time $t \geq 0$ with PDF $f(t) > 0$ and CDF $F(t)$: i.e.,

$$F(t) = \Pr(T \leq t) = \int_0^t f(x)dx.$$

We also define the **survival distribution** $S(t)$, which indicates the probability that the event has *not* occurred by time t:

$$S(t) = 1 - F(t) = \Pr(T > t) = \int_t^\infty f(x)dx.$$

The **hazard rate**[1] function $h(t)$ describes the instantaneous rate of occurrence of the event:

$$h(t) = \lim_{\delta \to 0} \frac{\Pr[t < T \leq t + \delta \mid T > t]}{\delta}.$$

The numerator in this expression is the probability that the event will occur in the interval $(t, t+\delta]$, *given that it has not occurred by time t*, and the denominator is the width of the interval. Hence, this fraction describes the rate of occurrence of the event, per unit of time. Taking the limit as $\delta \to 0$ yields the instantaneous rate of occurrence.

The hazard rate function can be simplified as follows:

$$\begin{aligned}
h(t) &= \lim_{\delta \to 0} \frac{\Pr[t < T \leq t + \delta \mid T > t]}{\delta} \\
&= \lim_{\delta \to 0} \left(\frac{\Pr[t < T \leq t + \delta, T > t]}{\delta} \right) \left(\frac{1}{\Pr[T > t]} \right) \\
&= \lim_{\delta \to 0} \left(\frac{\Pr[t < T \leq t + \delta]}{\delta} \right) \left(\frac{1}{\Pr[T > t]} \right) \\
&= \frac{f(t)}{S(t)} \\
&= \frac{f(t)}{1 - F(t)}.
\end{aligned}$$

This latter form often rears its[2] head in auction analyses.

[1] Also called a failure rate

[2] no-longer-seeming-so-ugly

Usually, a hazard rate is assumed to be increasing, decreasing, or constant. An increasing hazard rate signifies that the unit is becoming more and more prone to failure, with time. A decreasing hazard rate means the opposite: the unit is improving with time. Other possibilities include a U-shaped, or an upside-down U-shaped, hazard rate. The former is often used to model a human life span, because early in life we are very vulnerable, while at mid-life risks level off, until later in life when we become vulnerable again.

2 *Regular and MHR Distributions*

Observe that the virtual value function of a distribution F relates to the *inverse* hazard rate function h as follows:

$$\begin{aligned} \varphi(v) &= v - \frac{1 - F(v)}{f(v)} \\ &= v - \frac{1}{h(v)}. \end{aligned}$$

Now recall that Myerson's optimal auction design recipe requires that the virtual value function be weakly increasing in values: i.e., for $v \geq t \in T$, $\varphi(v) \geq \varphi(t)$. Distributions for which the corresponding virtual value function satisfies this property are called **regular**.

A related, and stronger, condition is that the hazard rate function be weakly increasing: i.e., for $v \geq t \in T$, $h(v) \geq h(t)$. This condition is called the **monotone hazard rate** (MHR) condition. Many common distributions satisfy the MHR condition: e.g., the uniform, the normal, and the exponential distributions.

MHR implies regularity, but the two are not equivalent (example forthcoming). One way to see that the MHR condition is stronger than regularity is the following: whereas regularity means the virtual value function is weakly increasing, MHR implies the virtual value function is *strictly* increasing, by the following reasoning. For $\delta > 0$,

$$\begin{aligned} h(v+\delta) &\geq h(v) && \text{MHR} \\ -\frac{1}{h(v+\delta)} &\geq -\frac{1}{h(v)} && \iff \\ v - \frac{1}{h(v+\delta)} &\geq v - \frac{1}{h(v)} && \implies, \text{ since } \delta > 0 \\ v + \delta - \frac{1}{h(v+\delta)} &> v - \frac{1}{h(v)} && \iff \\ \varphi(v+\delta) &> \varphi(v) && \text{virtual values are strictly increasing} \end{aligned}$$

Thus, MHR implies the virtual value function is *strictly* increasing. MHR also implies regularity.

Proposition 2.1. *MHR implies regularity.*

Proof. If the hazard rate $h(v)$ is weakly increasing in values, then the inverse hazard rate $1/h(v)$ is weakly decreasing in values: for $v \geq t$,

$$\frac{1-F(v)}{f(v)} \leq \frac{1-F(t)}{f(t)}.$$

Likewise,

$$-\frac{1-F(v)}{f(v)} \geq -\frac{1-F(t)}{f(t)}.$$

The virtual value function is thus weakly increasing in values:

$$v - \frac{1-F(v)}{f(v)} \geq v - \frac{1-F(t)}{f(t)} \geq t - \frac{1-F(t)}{f(t)}.$$

Therefore, MHR implies regularity. □

The MHR condition can be interpreted to mean that a distribution is not "heavy tailed." The PDF of a heavy-tailed distribution falls off more slowly than the PDF of the exponential distribution.

Finally, to show that the sets of MHR and regular distributions are distinct, we present an example of a distribution that satisfies regularity, but not MHR. This distribution indeed has heavy tails.

Example 2.2 (Regular, and not MHR)**.** The distribution

$$F(v) = 1 - \frac{1}{v+1}$$

has density

$$f(v) = \frac{1}{(v+1)^2},$$

and support $(0, \infty)$. The PDF and CDF are shown in Figure 1.

The hazard rate function is

$$\begin{aligned} h(v) &= \frac{f(v)}{1-F(v)} \\ &= \frac{\frac{1}{(v+1)^2}}{1 - \left(1 - \frac{1}{v+1}\right)} \\ &= \frac{1}{v+1}. \end{aligned}$$

Since $h(v)$ is a strictly decreasing function, F does not satisfy the MHR condition. The virtual value function, however, is a weakly increasing function:

$$\begin{aligned} \varphi(v) &= v - \frac{1}{h(v)} \\ &= v - (v+1) \\ &= -1. \end{aligned}$$

Therefore, F satisfies the regularity, but not the MHR, condition.

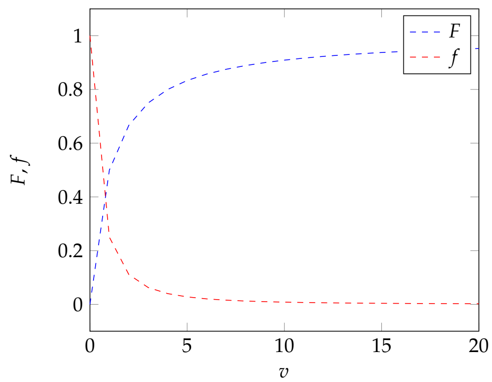

Figure 1: The PDF and CDF of a heavy-tailed distribution. At $v = 20$, $F(20) = .95$. At $v = 100$, $F(100) = .99$. At $v = 1000$, $F(1000) = .999$. This is an instance of the Burr distribution, $F(v, c, k) = 1 - (1 + v^c)^{-k}$, where $c = k = 1$.

Although F is regular, Myerson's scheme for allocating only to bidders with non-negative virtual values would not maximize revenue in an auction where values are distributed according to F, as no good would ever be allocated, so revenue would always be 0.

The issue is that F's support is unbounded, so Tonelli's theorem (which we invoked to swap the order of integration in our proof of Myerson's theorem) does not apply. Nonetheless, running a first- or second-price auction would generate positive expected revenue.