

# Applications of Myerson's Lemma

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We apply Myerson's lemma to solve the single-good auction, and the generalization in which there are  $k$  homogeneous goods: i.e.,  $k$  identical copies of the good. Our objective is welfare maximization.

## 1 Welfare-Maximizing Auctions

We can interpret Myerson's lemma as a recipe for designing an dominant-strategy incentive compatible (DSIC) and individually rational (IR) welfare-maximizing auction. The first step is to construct an (computationally) efficient feasible allocation rule that is monotonic in bidders' values, and the second, is to plug that rule into Myerson's payment formula to guarantee the incentive properties. When the allocation rule also achieves economic efficiency—meaning it optimizes (or approximately optimizes) welfare—we say that the auction is solved (or approximately solved).

## 2 Single-Good Auction

Our first application of Myerson's lemma is a simple sanity check. We have already discussed a DSIC auction design for the single-parameter setting with one good: the second-price auction, in which the highest bidder wins and pays the second-highest bid. Here, we confirm that Myerson's lemma leads us to the same conclusion.

Assume a single good auction with  $n$  bidders, each with a private value  $v_i$  in the range  $T_i = [0, \bar{v}_i]$ . (For simplicity, we assume  $\underline{v}_i = 0$ , for all bidders  $i \in [n]$ .)

*Welfare Maximization* Recall that welfare is the quantity  $\sum_i v_i x_i(\mathbf{v})$ , assuming ex-post feasibility: i.e.,  $\mathbf{x} \in \{0, 1\}^n$  and  $\|\mathbf{x}\| \leq 1$ . This quantity is maximized by awarding the good to a bidder with the highest value: i.e., a bidder  $i^*$  s.t.

$$i^* \in \arg \max_i v_i,$$

*Monotonicity* Fix a bidder  $i$  and a profile  $\mathbf{v}_{-i}$ . The necessary and sufficient condition for  $i$  to be allocated is that she bid higher than the highest bid among bidders other than  $i$ . This bid, which we denote  $b^*$ , is called the **critical bid**:  $b^* \equiv \max_{j \neq i} v_j =$  2nd-highest bid.

This allocation rule, which is monotonically weakly increasing in bidder  $i$ 's values, is plotted in Figure 1.

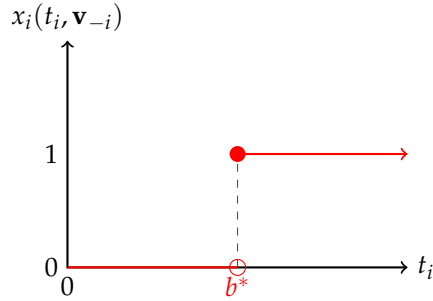


Figure 1: Bidder  $i$ 's allocation rule, for a fixed  $\mathbf{v}_{-i}$ . This rule is monotone.

**Claim** This allocation rule is monotonically weakly increasing.

*Proof.* If  $v_i < b^*$ , then  $x_i(v_i, \mathbf{v}_{-i})$  is 0, so increasing  $i$ 's bid cannot possibly lower  $i$ 's allocation. On the other hand, if  $v_i \geq b^*$  is a winning bid, so that  $x_i(v_i, \mathbf{v}_{-i})$  is 1, then  $x_i(v_i + \epsilon, \mathbf{v}_{-i})$  still equals 1, for all  $\epsilon > 0$ . In sum, for all  $v_i \in T_i$  and for all  $\epsilon > 0$ ,  $x_i(v_i + \epsilon, \mathbf{v}_{-i}) \geq x_i(v_i, \mathbf{v}_{-i})$ : i.e.,  $x_i$  is monotonically weakly increasing in values.  $\square$

*Payments* Assuming  $u_i(v_i, \mathbf{v}_{-i}) \geq 0$ , for all bidders  $i \in [n]$ , which ensures IR, the payment formula in turn ensures that only the winner of the auction will make a payment to the auctioneer. Assuming bidder  $i$  is a winner, her payment is as follows:

$$\begin{aligned} p_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i}) dz, \\ &= v_i \cdot 1 - \left[ \int_0^{b^*} 0 dz + \int_{b^*}^{v_i} 1 dz \right] \\ &= v_i - (v_i - b^*) \\ &= b^*. \end{aligned}$$

We split up the integral in this way because the allocation for bidding less than  $b^*$  bid is 0, while the allocation for bidding more is 1. Indeed, the payment  $b^*$  is the second-highest price. This payment is the region shaded in red in Figure 2.

We conclude that the combination of an allocating to a highest bidder together with charging the winner of the auction the second-highest bid is IC and IR. Since this allocation rule is economically and computationally efficient, this auction is solved.

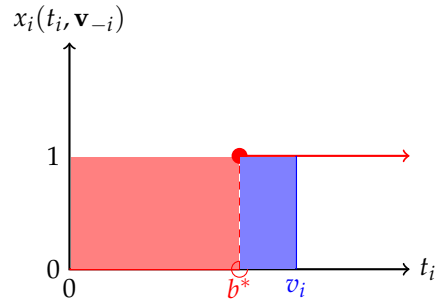


Figure 2: Bidder  $i$ 's payment function, for a fixed  $\mathbf{v}_{-i}$ . Payment is shown in red, and utility, for  $v_i$  as shown, in blue.

### 3 $k$ -Good Auction

Now consider an auction with  $k \geq 1$  identical copies of a good and  $n \geq k$  bidders, each with a private value  $v_i$  for a single copy of the good in the range  $T_i = [0, \bar{v}_i]$ .

*Welfare Maximization Problem* Generalizing the single-good case, welfare is the quantity  $\sum_i v_i x_i(\mathbf{v})$ , assuming ex-post feasibility: i.e.,  $\mathbf{x} \in \{0, 1\}^n$  and  $\|\mathbf{x}\| \leq k$ . This quantity is maximized by awarding the goods to the  $k$  highest bidders: i.e., by setting those entries of  $\mathbf{x}$  that correspond to the  $k$  largest bids to 1, and all others to 0.

*Monotonicity* Fix a bidder  $i$  and a profile  $\mathbf{v}_{-i}$ . The necessary and sufficient condition for  $i$  to be allocated is that her bid be among the  $k$  highest bids. The highest bid among those bidders who are not among the top  $k$ -highest bidders, which we denote  $b^*$ , is called the **critical bid**:  $b^* \equiv k\text{th-highest}_{j \neq i} v_j = k + 1\text{st-highest bid}$ .

Since the condition for being allocated is the same as it was in the single-good case (i.e., bidding above the critical bid), the allocation rule is the same as it was in the single-good case. This allocation rule, which is monotonically weakly increasing in bidder  $i$ 's values, is plotted in Figure 3.

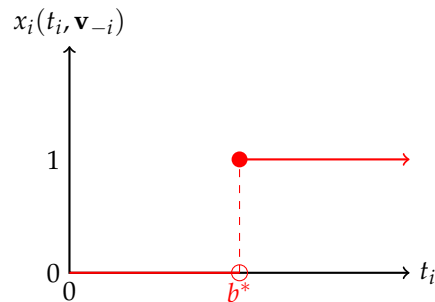


Figure 3: Bidder  $i$ 's allocation rule, for a fixed  $\mathbf{v}_{-i}$ . This rule is monotone.

**Claim** This allocation rule is monotonically weakly increasing.

*Proof.* If  $v_i < b^*$ , then  $x_i(v_i, \mathbf{v}_{-i})$  is 0, so increasing  $i$ 's bid cannot possibly lower  $i$ 's allocation. On the other hand, if  $v_i \geq b^*$  is a winning bid, so that  $x_i(v_i, \mathbf{v}_{-i})$  is 1, then  $x_i(v_i + \epsilon, \mathbf{v}_{-i})$  still equals 1, for all  $\epsilon > 0$ . In sum, for all  $v_i \in T_i$  and for all  $\epsilon > 0$ ,  $x_i(v_i + \epsilon, \mathbf{v}_{-i}) \geq x_i(v_i, \mathbf{v}_{-i})$ : i.e.,  $x_i$  is monotonically weakly increasing in values.  $\square$

*Payments* By the payment formula, if  $x_i = 0$ , then  $p_i = 0$ . Therefore, only the winners of the auction make a payment to the auctioneer. Assuming bidder  $i$  is a winner, her payment is as follows:

$$\begin{aligned} p_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i}) dz, \\ &= v_i \cdot 1 - \left[ \int_0^{b^*} 0 dz + \int_{b^*}^{v_i} 1 dz \right] \\ &= v_i - (v_i - b^*) \\ &= b^*. \end{aligned}$$

Since the condition for being allocated is the same as it was in the single-good case—simply bidding higher than  $b^*$ —this payment calculation is the same as it was in the single-good case. The payment  $b^*$  is the  $k + 1$ st-highest price; hence, this auction for  $k$  goods generalizes the second-price auction for a single good. This payment is the region shaded in red in Figure 4.

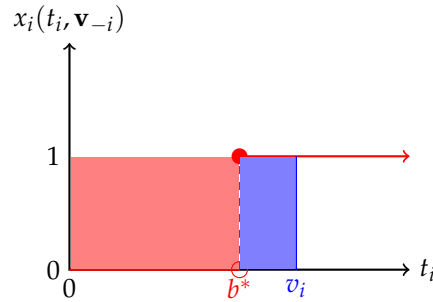


Figure 4: Bidder  $i$ 's payment function, for a fixed  $\mathbf{v}_{-i}$ . Payment is shown in red, and utility, for  $v_i$  as shown, in blue.

We conclude that the combination of allocating to the  $k$  highest bidders together with charging the winners of the auction the  $k$ th-highest bid is IC and IR. Since this allocation rule is economically and computationally efficient (simply sort the bids, and allocate to the top  $k$ ), this auction is solved. This solution is called the  $k$ -Vickrey auction.

*A two-good example.* Imagine three bidders,  $b_1, b_2$  and  $b_3$ , and two goods. The bidders' type spaces are closed intervals, but with different bounds: bidder  $i$ 's value lies in the range  $[0, i]$ .

Let  $v_i$  represent bidder  $i$ 's realized value. Suppose  $v_1 = 5/6$ ,  $v_2 = 2$ , and  $v_3 = 7/4$ . What happens in this example in the welfare-maximizing auction, IC, IR, and ex-post feasible auction?

To answer this question, we do the following:

1. Sort the bidders' values.
2. Find the winners: i.e., the bidders with the two highest values.
3. Determine the critical bid, and hence the winners' payments.

$i$	$v_i$	RANK	WINNER?	CRITICAL BID	PAYMENT
1	$5/6$	3	NO	N/A	N/A
2	2	1	YES	$5/6$	$5/6$
3	$7/4$	2	YES	$5/6$	$5/6$

Table 1: Example Two-Good Auction

These steps are illustrated in Table 1. Bidders 2 and 3 are allocated the goods, as they have the two highest values. They each pay the critical bid, which in this two-good auction is the third-highest value.