An Application of Myerson's Theorem

CSCI 1440/2440

2023-02-22

We apply Myerson's theorem to solve the k-good auction, where there are k identical copies of a good. Our objective is revenue maximization.

1 *Revenue-Maximizing Auctions*

Myerson's theorem equates virtual welfare with revenue, so together with Myerson's lemma, we have a recipe for designing DSIC and IR revenue-maximizing auctions. The first step is to construct an (computationally) efficient feasible allocation rule that is monotonic in virtual values, and the second step is to plug that rule into Myerson's payment formula to guarantee the incentive properties. When that allocation rule also achieves economic efficiency—meaning it optimizes (or approximately optimizes) virtual welfare, and hence revenue—we say that the auction is solved (or approximately solved).

2 *k-Good Auction*

Assume there are $k \geq 1$ identical copies of a good and $n \geq k$ bidders, each with a private value v_i for a single copy of the good drawn from a *regular* distribution F_i with bounded support $T_i = [0, \overline{v}_i]$. (Once again, for simplicity we assume $\underline{v}_i = 0$, for all bidders $i \in [n]$.)

Allocation By Myerson's theorem, to maximize revenue it suffices to allocate so as to maximize virtual welfare. We proceed as follows:

- Sort bidders by *virtual* value, so that $\varphi_1(v_1) \leq \varphi_2(v_2) \leq \cdots \leq \varphi_{n-1}(v_{n-1}) \leq \varphi_n(v_n)$.
- Allocate nothing to any bidders with *negative* virtual values, or to the lowest $n-k$ bidders.
- Among the remaining top $m \leq n$ bidders with *non-negative* virtual values, assign bidder $m+j-1$ good $k-j+1$, for $1 \leq j \leq k$.
  - m gets good k
  - $m+1$ gets good $k-1$
  - $m+2$ gets good $k-2$
  - $\vdots$
  - $m+k-1$ gets good 1

Remark 2.1. Although $n \geq k$, it is nonetheless possible that $m < k$, if ever fewer than k bidders have non-negative virtual values. In such cases, only m goods are assigned.

This allocation rule is feasible; moreover, it optimizes virtual welfare, and hence, by Myerson's theorem, revenue. Next, we argue that it is also weakly monotonic; specifically, weakly increasing in values.

Monotonicity Fix a bidder i and a profile $\mathbf{v}_{-i}$. Define φ^* as the kth-highest virtual value among bidders other than i:
$\varphi^* \equiv k\text{th-highest}_{j \in N \setminus \{i\}} \varphi_j(v_j) = k + 1\text{st-highest virtual value.}$

Case 1 Assuming $\varphi^* \geq 0$, the necessary and sufficient condition for i to be allocated is that she bid b s.t.

$$\varphi_i(b) \geq \varphi^*;$$

equivalently,

$$b \geq \varphi_i^{-1}(\varphi^*).$$

Case 2 If $\varphi^* < 0$, then bidder i need not outbid anyone; she need only bid enough so that their own virtual value is non-negative: i.e.,

$$\varphi_i(b) \geq 0;$$

equivalently,

$$b \geq \varphi_i^{-1}(0).$$

The value $\varphi_i^{-1}(0)$ is called bidder i's **reserve** price. In summary, bidder i is a potential winner iff $\varphi_i(b) \geq \max\{\varphi^*, 0\}$.

This allocation rule can be summarized as follows: for $b \in T_i$,

$$x_i(b, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } \varphi_i(b) \geq 0 > \varphi^* \\ 1 & \text{if } \varphi_i(b) > \varphi^* \geq 0 \\ ? & \text{if } \varphi_i(b) = \varphi^* \geq 0 \\ 0 & \text{if } \varphi_i(b) < \varphi^* \end{cases} \tag{1}$$

That is, i is the sole winner if $\varphi_i(b) \geq 0$ and $\varphi_i(b) > \varphi^*$, while i is a loser if $\varphi_i(b) < \varphi^*$. On the other hand, if $\varphi_i(b) \geq 0$ and there is a tie for the highest virtual value, then the allocation is as-of-yet unspecified. We claim that this allocation rule is weakly monotonic in values, assuming either of the following tie-breaking strategies:[1]

[1] Proofs are provided in Appendix A.

- If the bidders' value distributions are MHR, so that virtual values are *strictly* increasing in value, then ties can be broken arbitrarily.
- If the bidders' value distributions are regular, so that virtual values are *weakly* increasing in value, then ties must be broken deterministically, meaning according to some pre-specified rule, such as lexicographically (e.g., in alphabetical order by bidders' names).

Payments By individual rationality, if $x_i = 0$, then $p_i = 0$. Therefore, only the winners of the auction make a payment to the auctioneer. Letting b^* denote bidder i's **critical bid**, above which bidder i is allocated, bidder i's payment is as follows:

$$
\begin{aligned}
p_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, \mathrm{d}z, \\
&= v_i \cdot 1 - \left[\int_0^{b^*} 0\, \mathrm{d}z + \int_{b^*}^{v_i} 1\, \mathrm{d}z \right] \\
&= v_i - (v_i - b^*) \\
&= b^*.
\end{aligned}
$$

Payments in the revenue-maximizing auction are syntactically equivalent to the payments in the welfare-maximizing auction. They differ, however, in the meaning of b^*. Whereas b^* was the $k+1$st-highest bid in the welfare-maximizing case, here it is the inverse, according to i's virtual value function, of φ^* (the kth-highest virtual value among bidders other than i), assuming φ^* is non-negative; or, if φ^* is negative, it is the inverse, according to i's virtual value function, of 0. In sum, payments are given by:

$$
p_i(v_i, \mathbf{v}_{-i}) = \begin{cases} \varphi_i^{-1}(\varphi^*) & \text{if } \varphi^* \geq 0 \\ \varphi_i^{-1}(0) & \text{otherwise} \end{cases}
$$

N.B. Assuming regularity, if $\varphi_i^{-1}(\varphi^*) \geq \varphi_i^{-1}(0)$ then $\varphi^* \geq 0$, so that $p_i(v_i, \mathbf{v}_{-i}) = \varphi_i^{-1}(\varphi^*)$. Otherwise, $p_i(v_i, \mathbf{v}_{-i}) = \varphi_i^{-1}(0)$.

In summary, in the optimal (i.e., revenue-maximizing) k-good auction, bidder i is a potential winner iff $\varphi_i(b) \geq \max\{\varphi^*, 0\}$, and any winning bidder i pays $p_i(v_i, \mathbf{v}_{-i}) = \max\{\varphi_i^{-1}(\varphi_i^*), \varphi_i^{-1}(0)\}$.

3 *A Revenue-Maximizing Two-Good Auction*

Imagine three bidders, b_1, b_2 and b_3, and two goods. The bidders' values are uniformly distributed on closed intervals, but with different bounds: each bidder i's value is uniformly distributed on the closed interval $[0, i]$, so $f_i(v) = 1/i$, and $F_i(v) = v/i$, for all $v \in [0, i]$. Let v_i represent bidder i's realized value. Suppose $v_1 = 5/6$, $v_2 = 2$, and $v_3 = 7/4$. What happens in this example in the revenue-maximizing, IC, IR, and ex-post feasible auction?

To answer this question, we do the following:

1. Calculate the virtual value function for each bidder.

2. Find each bidder's virtual value.

3. Sort the virtual values.

4. Throw out the bidders with negative virtual values.
5. Among the remaining bidders, find the winners: i.e., the bidders with the highest virtual value.
6. Determine each winner's critical bid, and hence each winner's payment.

Table 1: Example Two-Good Auction

i	v_i	$f_i(v)$	$F_i(v)$	$\varphi_i(v)$	$\varphi_i(v_i)$	Rank	$\varphi_i(v_i) \geq 0$?	Winner?	Critical bid	Payment
1	$5/6$	1	v	$2v-1$	$2/3$	2	YES	YES	$1/2$	$\varphi_1^{-1}(1/2) = 3/4$
2	2	$1/2$	$v/2$	$2v-2$	2	1	YES	YES	$1/2$	$\varphi_2^{-1}(1/2) = 5/4$
3	$7/4$	$1/3$	$v/3$	$2v-3$	$1/2$	3	YES	NO	N/A	N/A

These steps are illustrated in Table 1. Bidders 1 and 2 are allocated the goods, because they have the two highest virtual values, and neither of their virtual values are negative. They each pay the inverse of their virtual value function at their critical bid, which in this example is the third-highest virtual value, since that value is not negative. In Table 2, the third-highest virtual value *is* negative, so the winning bidders pay the inverse of their virtual value function at 0. Observe that in both examples, bidder 3's value is higher than bidder 1's; however, bidder 3's virtual value is lower than bidder 1's. So bidder 1 is allocated, while bidder 3 is not.

Table 2: Example Two-Good Auction

i	v_i	$f_i(v)$	$F_i(v)$	$\varphi_i(v)$	$\varphi_i(v_i)$	Rank	$\varphi_i(v_i) \geq 0$?	Winner?	Critical bid	Payment
1	$5/6$	1	v	$2v-1$	$2/3$	2	YES	YES	0	$\varphi_1^{-1}(0) = 1/2$
2	2	$1/2$	$v/2$	$2v-2$	2	1	YES	YES	0	$\varphi_2^{-1}(0) = 1$
3	1	$1/3$	$v/3$	$2v-3$	-1	3	NO	NO	N/A	N/A

A Monotonicity in the Face of Ties

Ties can create difficulties, because allocations are a function of *virtual* values, not values. As a result, an allocation rule that breaks ties (among virtual values) arbitrarily might declare bidder i a winner when their value is v_i, but a loser when their value is $t_i > v_i$, thereby violating monotonicity. This cannot happen when virtual values are strictly increasing in values (MHR); however, it can happen when virtual values are weakly increasing in values (regular).

MHR Distributions. If the bidders' value distributions are MHR, then their virtual values are strictly increasing in values, by definition. In this case, an increase in value can only increase a bidder's virtual value, thereby necessarily breaking any existing ties in their favor, and making it more likely that she is allocated.

Proposition A.1. *If the bidders' value distributions are MHR, then the allocation rule given by Equation 1 is monotonically increasing in values, regardless of the choice of tie-breaking rule.*

Proof. Let $\varphi^* = k\text{th-highest}_{j \in [n] \setminus \{i\}} \varphi_j(v_j)$. If $\varphi_i(b) < \varphi^*$, then b is a losing bid (i.e., $x_i(b, \mathbf{v}_{-i}) = 0$), so increasing b cannot possibly lower i's allocation. Indeed, for all losing bids $b \in T_i$ and $\epsilon > 0$ s.t. $b + \epsilon \in T_i$, $x_i(b + \epsilon, \mathbf{v}_{-i}) \geq x_i(b, \mathbf{v}_{-i})$.

On the other hand, if $\varphi_i(b) \geq \varphi^*$, then b is potentially a winning bid (depending on the tie-breaking rule). By MHR, however, $\forall \epsilon \geq 0$ s.t. $b + \epsilon \in T_i$, $\varphi_i(b + \epsilon) > \varphi^*$, so that $x_i(b + \epsilon, \mathbf{v}_{-i}) = 1$. Indeed, for all potentially winning bids $b \in T_i$ and $\epsilon > 0$ s.t. $b + \epsilon \in T_i$, $x_i(b + \epsilon, \mathbf{v}_{-i}) \geq x_i(b, \mathbf{v}_{-i})$. □

Regular Distributions If a bidder's value distribution is regular, but not MHR, then an increase her value need not lead to an increase in her virtual value; her virtual value could remain constant. This phenomenon could pose a problem in the case of ties: i.e., two (or more) bidders with the same virtual value.

If there is a tie, and a bidder's value increases but her virtual value does not, and if ties are not broken carefully (e.g., if they are broken arbitrarily), it is possible that a bidder goes from being allocated at a lower value to not being allocated at a higher value. To rule out this possibility, we assume a *deterministic* tie-breaking rule when bidders' value distributions are regular—one which guarantees weak monotonicity of the allocation rule *ex-post*, and by linearity of expectations, interim and *ex-ante* as well.

Proposition A.2. *If the distribution from which types are drawn is regular, and the tie-breaking rule is deterministic and weakly monotonic, then the allocation rule given by Equation 1 is weakly monotonic in values.*

Proof. As in the case of MHR distributions, increasing a losing bid cannot possibly lower a bidder's allocation. In the case of a potentially winning bid, however, where $\varphi_i(b) \geq \max_{j \in [n] \setminus \{i\}} \varphi_j(v_j)$, it is possible that $x_i(b, \mathbf{v}_{-i}) = 1$, while $x_i(b + \epsilon, \mathbf{v}_{-i}) = 0$, for some $\epsilon > 0$ s.t. $b + \epsilon \in T_i$. If $x_i(b, \mathbf{v}_{-i}) = 1$, then ties were broken in i's favor at bid b. A deterministic tie-breaking rule is necessary to ensure that ties are again broken in i's favor again at bid $b + \epsilon$. Assuming such a rule, $x_i(b + \epsilon, \mathbf{v}_{-i}) \geq x_i(b, \mathbf{v}_{-i})$. □

Taking expectations over ex-post allocations w.r.t. $\mathbf{v}_{-i}$ ($\mathbf{v}$) yields a deterministic interim (*ex-ante*) allocation rule, which by linearity of expectations and Proposition A.2, is weakly monotonic in values.