Bayes-Nash Equilibrium in the First-Price Auction

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We state and prove a Bayes-Nash Equilibrium strategy for the first-price auction, assuming the bidders' values are drawn i.i.d. from the uniform distribution on $[0, 1]$.

1 *The First-Price, Sealed-Bid Auction*

A first-price auction is an example of a *pay-your-bid* auction. In this auction format, whoever submits the highest bid is the winner, and she pays what she bid, namely the highest bid. Ties are broken randomly: if multiple bidders submit the highest bid, exactly one of them is chosen as the winner.

Theorem 1.1. *In a first-price auction with bidders $i \in [n]$, if all bidders' values v_i are drawn i.i.d. from the uniform distribution on $[0, 1]$, then the bidding strategies $b_i = \left(\frac{n-1}{n}\right) v_i$ comprise a Bayes-Nash equilibrium.*

Proof. Fix a bidder i. We assume that all bidders besides i bid according to this formula, and argue that bidder i should do the same.

Let z represent i's bid. There are two possible outcomes:

- Case 1: Someone outbids i: i.e., there exists a bidder $j \neq i$ s.t. $\left(\frac{n-1}{n}\right) v_j > z$. In this case, i does not win the good, so $u_i = 0$.
- Case 2: No one outbids i: i.e., for all bidders $j \neq i$, $z \geq \left(\frac{n-1}{n}\right) v_j$.

  In this case, i wins the good,[1] so $u_i = v_i - z$.

[1] We assume ties are broken in i's favor.

Bidder i's expected utility is equal to the probability of Case 1 times the utility it earns in Case 1 plus the probability of Case 2 times the utility it earns in Case 2. As the utility earned in Case 1 is zero, we need only concern ourselves with the probability of Case 2.

The probability of this latter event is:

$$\Pr\left(z \geq \left(\frac{n-1}{n}\right) v_j, \text{for all bidders } j \neq i\right) = \Pr\left(v_j \leq \frac{nz}{n-1}, \text{for all bidders } j \neq i\right) \tag{1}$$

$$= \prod_{j \neq i} \Pr\left(v_j \leq \frac{nz}{n-1}\right) \tag{2}$$

$$= \left(F\left(\frac{nz}{n-1}\right)\right)^{n-1} \tag{3}$$

$$= \left(\frac{nz}{n-1}\right)^{n-1} \tag{4}$$

Equation 1 follows via algebra. Equation 2 follows from the fact that the values are drawn i.i.d. Equation 3 is the definition of a CDF. Finally, Equation **??** follows because F is the CDF of a uniform distribution.

Bidder i's expected utility is thus:

$$\begin{aligned} \mathbb{E}[u_i] &= \underbrace{\left(\frac{nz}{n-1}\right)^{n-1}(v_i - z)}_{i \text{ wins}} + \underbrace{\left(1 - \left(\frac{nz}{n-1}\right)^{n-1}\right) \cdot 0}_{i \text{ loses}} \\ &= \left(\frac{n}{n-1}\right)^{n-1} z^{n-1}(v_i - z). \end{aligned}$$

Next, we take the derivative of $\mathbb{E}[u_i]$ with respect to z, and set it equal to 0, to maximize i's expected utility. Since $\left(\frac{n}{n-1}\right)^{n-1}$ is a just a constant, it eventually drops out, so we drop it from the start:

$$\frac{d}{dz}\mathbb{E}[u_i] = \frac{d}{dz}\left[z^{n-1}(v_i - z)\right] = (n-1)z^{n-2}(v_i - z) - z^{n-1}$$

Setting this derative equal to zero yields:

$$\begin{aligned} \frac{d}{dz}\mathbb{E}[u_i] &= 0 \\ (n-1)z^{n-2}(v_i - z) - z^{n-1} &= 0 \\ (n-1)(v_i - z) - z &= 0 \\ (n-1)v_i - nz &= 0 \end{aligned}$$

Therefore, bidder i maximizes her utility by bidding:

$$z = \left(\frac{n-1}{n}\right) v_i,$$

so that the given bidding strategy is indeed a Bayes-Nash equilibrium in a first-price auction under the stated assumptions.

Technical Note. Since a bid of $\frac{n-1}{n}$ guarantees that i wins the auction, it sufficies to restrict z to lie in the range $[0, \frac{n-1}{n}]$. A rigorous proof would note that while $z = \left(\frac{n-1}{n}\right) v_i$ yields positive utility, neither of the two extreme points, $z = 0$ nor $z = \frac{n-1}{n}$, do; and would also verify that the second derivative of $\mathbb{E}[u_i]$ is negative at $z = \left(\frac{n-1}{n}\right) v_i$. We leave this final step as an exercise for the reader. □