Homework 6: The Vickrey-Clarke-Groves Mechanism

CSCI 1440/2440

2025-10-23

Due Date: Tuesday, October 28, 2025. 11:59 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit via Gradescope with you and your partner's Banner ID's and which course you are taking.

For 1000-level credit, you need to solve the first four problems. For 2000-level credit, you should solve all five problems.

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1 Payment Bounds

The VCG mechanism collects (possibly misleading) reports from all bidders and then computes a welfare-maximizing allocation ω^* together with corresponding payments $p_i(\omega^*)$, which ensure that a dominant strategy comprises truthful reports.

Prove that, for all bidders $i \in [n]$,

1. VCG payments $p_i(\omega^*)$ are lower-bounded by 0.
2. VCG payments $p_i(\omega^*)$ are upper-bounded by $v_i(\omega^*)$.

2 Revenue

This problem asks you to construct examples of multiparameter auctions, by which we mean you should specify 1. a set of goods, and 2. each bidder's valuation. Let $\text{Rev}([n])$ be the revenue generated by the VCG mechanism, given a set of bidders $[n]$.

1. Show, by example, that the following is possible: $\text{Rev}([n] \setminus \{i\}) > \text{Rev}([n])$, for some $i \in [n]$.
2. Construct a non-trivial example of a multiparameter auction in which the VCG mechanism yields $\text{Rev}([n]) = 0$. (By non-trivial, we mean that each bidder's valuation should be non-negative for every outcome and strictly positive for at least one outcome, and the number of bidders should exceed 1.)

3 Collusion

This problem highlights two shortcomings of the VCG mechanism.

1. Show that, when the VCG mechanism is used, it is possible for two bidders not to be allocated anything when bidding truthfully, but for one or both of them to obtain strictly positive utility if they tacitly collude,[1] by both submitting untruthful reports.

[1] To collude tacitly is to collude without explicit communication.

2. Suppose that a single bidder submits multiple false bids to a VCG mechanism. That is, the auctioneer thinks there are n distinct bidders, but in reality, at least two of the n bids are submitted by a single bidder i. Show that a bidder can obtain strictly higher utility by submitting multiple false bids rather than one truthful bid.

4 *Ties*

The outcome, namely the allocation and payments, of a VCG mechanism depends on the solution to welfare-maximization problem, which may not be unique. Multiple optima would not impact the welfare, as if one of them yielded lower welfare it would not be welfare-maximizing! But a question arises as to whether the choice of welfare-maximizing solution (i.e., allocation) on which the outcome (i.e., payments) depend impacts revenue.

It turns out that the revenue earned in a VCG mechanism also does not depend on the choice of welfare-maximizing allocation. Prove this claim in the case of 2 goods and 2 bidders.

5 *Unit Demand*

A **unit-demand** valuation is one in which bidders value a bundle based on only the value of their favorite good in the bundle. More formally, let S denote the set of goods, and let v_{ij} represent bidder i's value for good $j \in S$. If bidder i has unit demand, then

$$v_i(S) = \max_{j \in S} v_{ij}, \quad \forall S \subseteq G. \tag{1}$$

Prove that the VCG mechanism can be run in time polynomial in the number of bidders and goods, assuming unit-demand valuations. In other words, prove that the allocation problem can be solved in polynomial time in this special case.

Hint: Use the Hungarian algorithm.