Homework 4: Myerson's Theorem

CSCI 1440/2440

2025-10-02

Due Date: Tuesday, October 7, 2025. 11:59 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit via Gradescope with you and your partner's Banner ID's and which course you are taking.

For 1000-level credit, you need only solve the first three problems. For 2000-level credit, you should solve all four problems.

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1 *Bayesian Constraints*

Recall that first-price, third-price, and all-pay auctions all employ the same allocation rule, namely, allocate the good to the highest bidder. In Homework 3, we transformed all three of these auctions to the same BIC auction via the revelation principle, as all three auctions transformed to one with the same payments. Therefore, these welfare-maximizing auctions all yield the same revenue, which is in fact the same revenue as that of the second-price (DSIC) auction.

Myerson's theorem tells us that optimal revenue can be understood as optimal virtual welfare in BIC auctions, and hence in DSIC auctions, since DSIC implies BIC. You might expect an optimal BIC auction to yield greater revenue than an optimal DSIC auction, since their objectives are the same, while BIC is a less constraining requirement than DSIC. This turns out not to be the case, however, assuming regularity (i.e., non-decreasing virtual value functions).

Prove that just as for welfare-maximizing auctions, the revenue achieved by a revenue-maximizing BIC auction does not exceed that of the revenue-maximizing DSIC auction, which allocates in order of virtual values, as long as bidders meet their reserve.

Hint: Use the following fact from convex optimization: an optimum of a linear function over a convex space (if it exists) is attained at an extreme point (i.e., one that cannot be written as a convex combination of others). And then argue the following: the space of BIC-feasible interim allocation rules is convex; expected revenue is linear in the interim allocation rule; the (pointwise) allocation rule "allocate in order of virtual values, assuming bidders meet their reserve" is an optimum in the convex space of BIC-feasible interim allocation rules. Finally, use Myerson's lemma and regularity to conclude that the auction defined by this allocation rule is DSIC, and thus a DSIC auction is revenue-maximizing under the BIC constraints.

2 *Revenue Equivalence?*

In this problem, you will explore the expected revenue of two different auction formats, assuming two bidders whose values are both drawn i.i.d. from a uniform $U[0,1]$ distribution.

1. What is the expected revenue of Myerson's optimal (i.e., revenue-maximizing) auction?

   Hints: What is each bidder i's virtual value function φ_i? Is this virtual value function weakly increasing in values (i.e., is $U[0,1]$ regular)? What is the inverse of this virtual value function, and what is the reserve price?

2. Derive the expected revenue of a second-price auction with this same reserve. In this auction, the highest bidder wins (as long as their bid exceeds the reserve) and pays the maximum of the second-highest bid and the reserve.

3. Comment on the expected revenue of these two auctions, and compare their revenue to the revenue of the second-price auction without reserve prices.

3 *Another Auction with Reserve Prices*

Reserve prices are necessary to maximize expected revenue, since they give auctioneers the flexibility to charge more money to bidders whom they expect to have high values, while still preserving incentive compatibility and individual rationality.

Amy understands this, but she does not understand why the revenue-maximizing auction for a single good would allocate to a bidder with the highest virtual value, rather than a bidder with the highest value. To her, it seems like the revenue-maximizing auction should allocate to a bidder with the highest value, since such a bidder is willing to pay the most!

So Amy proposes the following revision to Myerson's optimal auction: use the same personalized reserve prices as Myerson, so that bidders cannot win if they do not bid above their reserve prices, but allocate to a bidder with the highest value among those who do bid above their reserve prices. More precisely, Amy's allocation rule works as follows:

- Assume $v_i \sim F_i$ for all bidders $i \in [n]$, where F_i is regular with bounded support.

- Determine the set C of bidders i who bid at least their reserve, namely $\phi_i^{-1}(0)$, where ϕ_i is bidder i's virtual value function.
- Allocate the good to the highest bidder in C.

Amy also wants her auction to be IC and IR. Thus, she proposes using Myerson's payment characterization lemma to set payments.

1. Explain Amy's payment rule in words. What does the winning bidder pay?
2. When at most one bidder bids above her reserve prices, Amy's and Myerson's auctions yield the exact same outcome.

   Consider the case in which there are at least two bidders who bid above their reserve prices, and among them, the bidder with the highest value is not the same as the bidder with the highest virtual value. Prove that Amy's auction yields weakly greater expected revenue than Myerson's auction in this case.
3. Consider the other case, in which the bidder with the highest value also has the highest virtual value. In this case:
   (a) Provide an example in which Amy's auction produces greater revenue.
   (b) Provide an example in which Myerson's auction produces greater revenue.

   Your examples should define the number of bidders, each bidder's value distribution, and each bidder's value.
4. **Extra Credit:** Suppose there are just two bidders: Bidder A whose value is drawn from $U(0,1)$ and Bidder B whose value is drawn from $U(0,2)$. Compute the expected revenue of both Amy's and Myerson's auctions.

4 *A Simple(r) Approximately-Optimal Auction*

Assuming regularity (i.e., weakly increasing virtual values), prove that the expected revenue of the second-price auction with personalized reserve prices is always at least half of the expected revenue of the optimal auction. In this simple(r) auction, the highest bidder wins as long as her bid exceeds her reserve, and she pays the maximum of the second-highest bid and her reserve.

N.B. This second-price auction with monopoly reserves is simpler than Myerson's optimal auction, as it requires inverting each virtual value function only at 0, to compute reserve prices.

Hint: Here is an outline of how to proceed. Define OPT as the expected revenue of the optimal auction, and APX as the expected revenue of the simpler auction. The goal is to show APX $\geq$ 1/2 OPT.

Assume bidder i wins the optimal auction, and bidder j, the simple auction, so that $\text{OPT} = \mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i)]$ and $\text{APX} = \mathbb{E}_{v_j \sim F_j}[p_j]$. As the winners of the auctions, i and j, are random variables,

$$\text{OPT} = \underbrace{\mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \mid i = j] \Pr[i = j]}_{\text{the same winner}} + \underbrace{\mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \mid i \neq j] \Pr[i \neq j]}_{\text{different winners}}$$

Thus, it suffices to show (1) APX is at least the expected revenue of the optimal auction when $i = j$ (i.e., when the same bidder wins both auctions); and (2) APX is at least the expected revenue of the optimal auction when $i \neq j$ (i.e., when the winners of the two auctions differ): i.e.,

$$\text{APX} \geq \underbrace{\mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \mid i = j] \Pr[i = j]}_{\text{the same winner}}$$
$$\text{APX} \geq \underbrace{\mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \mid i \neq j] \Pr[i \neq j]}_{\text{different winners}},$$

from which it follows that 2APX $\geq$ OPT, because

$$\begin{aligned}\text{OPT} &= \mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i)] \\ &= \underbrace{\mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \mid i = j] \Pr[i = j]}_{\text{the same winner}} \\ &\quad + \underbrace{\mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \mid i \neq j] \Pr[i \neq j]}_{\text{different winners}}.\end{aligned}$$