Homework 5: Posted-Price Mechanisms

CSCI 1440/2440

2025-10-14

Due Date: Tuesday, October 21, 2025. 11:59 PM.

We encourage you to work in groups of size two. Each group need only submit one solution. Your submission must be typeset using LaTeX. Please submit via Gradescope with you and your partner's Banner ID's and which course you are taking.

For 1000-level credit, you need only solve the first four problems. For 2000-level credit, you should solve all five problems.

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1 *An Equal-Revenue Distribution*

Suppose all bidders draw their values for a single good from the following distribution:

$$F(v) = \begin{cases} 0 & \text{for } v \leq 1 \\ 1 - 1/v & \text{for } v \geq 1 \end{cases}$$

1. Assuming just one bidder, what revenue curve corresponds to this distribution, when the posted price is $\pi > 1$? **Hint:** Distributions of this form are called **equal-revenue distributions**.
2. What virtual value function corresponds to this distribution?
3. How would Myerson's optimal auction allocate this good? What would each bidder pay to satisfy IC and IR?
4. Why isn't Myerson's auction optimal in this setting?
5. Describe an IC and IR auction that achieves greater expected revenue than Myerson's "optimal" auction given this distribution.

2 *Revenue Curves and the Median*

Assuming a single buyer with regular distribution F, the revenue curve as a function of a quantile $q \in [0, 1]$ is given by:

$$\text{Rev}(q) = q\, F^{-1}(1 - q)\,.$$

1. What is the value of the revenue curve at quantile $q = 1/2$? State your answer in terms of the median, call it κ.

2. Prove that the median upper bounds the revenue curve: i.e., $\text{Rev}(q) \leq \kappa$, for all $q \in [0,1]$. **Hint:** Use the fact that the revenue curve is concave.

3. Extra credit: Can you derive a tighter upper bound for $q \in [0, 1/2]$ or $q \in [1/2, 1]$—in terms of the median κ?

3 *Optimizing Posted Prices*

Consider a setting in which one good is for sale, and there are n potential buyers/bidders whose values are all drawn i.i.d. from the distribution F.

1. When types are drawn i.i.d. from an arbitrary distribution F, the total expected revenue generated by a posted-price mechanism is:

$$\text{Rev}\,(n) = \pi \left(1 - F\,(\pi)^n\right). \tag{1}$$

   Solve for the optimal posted price for an arbitrary distribution F. Your solution should be in terms of $F(\pi)$ and $f(\pi)$.

2. The CDF and PDF of the first-order statistic is given by $F\,(\pi)^n$ and $nF\,(\pi)^{n-1}\, f\,(\pi)$, respectively. Using this information, relate the optimal posted price to the hazard rate of the first-order statistic distribution. **Reminder:** The hazard rate of a continuous random variable T with CDF F and PDF f is defined as $f(t)/1-F(t)$.

4 *Posted-Price vs. Second-Price Revenue*

Consider a symmetric setting comprising n bidders whose values are all drawn i.i.d. from the uniform distribution on $[0, 1]$.

Prove that the ratio of the expected revenue of the posted-price mechanism for a single good, with posted price $\pi = F^{-1}\,(1 - 1/n)$, to that of the second-price auction is:

$$\frac{\text{PP}}{\text{2nd}} \geq 1 - \frac{1}{e}$$

Hint: $(1 - 1/n)^n \leq 1/e$, for all $n \geq 1$.

5 *Approximately-Optimal Second-Price Auctions*

This problem concerns the **single-sample second-price auction**, a second-price auction with a reserve price, which works as follows:

Bids are collected. A reserve price is chosen by removing an arbitrary bidder j from the auction, and setting the reserve price to be j's bid. The auctioneer then allocates the good to the bidder with the highest bid iff their bid is at least this reserve, and charges the winner, if any, the greater of the second-highest bid and the reserve price.

The goal of this problem is to show that this auction achieves $\frac{1}{2}\left(\frac{n-1}{n}\right)$ of the optimal revenue, where n is the number of bidders, assuming symmetic values drawn i.i.d. from a regular distribution F.

1. Let t and v be two values in the support of F. Let r^* be the monopoly reserve price: i.e., $r^* = \varphi^{-1}(0)$. Show that

$$\mathop{\mathbb{E}}_{v \sim F}\left[\text{Rev}\left(\max\{t, v\}\right)\right] \geq 1/2 \; \text{Rev}\left(\max\{t, r^*\}\right).$$

   Hint: Prove this result in quantile space, namely:

$$\mathop{\mathbb{E}}_{q(v) \sim U(0,1)}\left[\text{Rev}\left(\min\{q(t), q(v)\}\right)\right] \geq 1/2 \; \text{Rev}\left(\min\{q(t), q(r^*)\}\right).$$

2. Let APX denote the expected revenue generated by the single-sample second-price auction. Let OPT denote the expected revenue generated by the optimal (i.e., revenue-maximizing) auction. Using part 1, show that the single-sample second-price auction generates, in expectation, approximately half the total expected revenue generated by the optimal auction: i.e.,

$$\frac{\text{APX}}{\text{OPT}} \geq \frac{1}{2}\left(\frac{n-1}{n}\right).$$