Homework 5: Myerson’s Theorem
CSCI 1440/2440
2023-03-21

Due Date: Tuesday, March 7, 2023. 11:59 PM.

We encourage you to work in groups of size two. Each group needs only submit one solution. Your submission must be typeset using \LaTeX. Please submit via Gradescope with you and your partner’s Banner ID’s and which course you are taking.

For 1000-level credit, you need only solve the first three problems. For 2000-level credit, you should solve all four problems.

1 Bayesian Constraints, continued

Recall the Bayesian (i.e., interim) formulation of the auction design problem from Homework 4. Since the interim constraints are weaker than the ex-post constraints presented in lecture, you might imagine that the welfare achieved by the welfare-maximizing auction in the interim case exceeds that of the ex-post case. Argue that this is not in fact the case: i.e., that the value that maximizes expected welfare in this model is the same regardless of whether the IC and IR constraints are interim or ex-post. (The same holds for revenue, but you need only make the argument once.)

2 Revenue Equivalence?

In this problem, you will explore the expected revenue of different auction formats. Answer the questions, assuming two bidders whose values are both drawn i.i.d. from a uniform $U[0, 1]$ distribution.

1. What is the expected revenue of Myerson’s optimal (i.e., revenue-maximizing) auction?

   **Hints:** What is each bidder $i$’s virtual value function $\phi_i$? Is this virtual value function weakly increasing in values (i.e., is $U[0, 1]$ regular)? What is the inverse of this virtual value function, and what is the reserve price?

2. Derive the expected revenue of a second-price auction with this same reserve. In this auction, the highest bidder wins (as long as their bid exceeds the reserve) and pays the maximum of the second-highest bid and the reserve.
3. Comment on the expected revenue of these two auctions, and compare their revenue to the revenue of the second-price auction without reserve prices.

3 Another Auction with Reserve Prices

Reserve prices are necessary to maximize expected revenue, since they give auctioneers the flexibility to charge more money to bidders whom they expect to have high values, while still preserving incentive compatibility and individual rationality.

Amy understands this, but she does not understand why the revenue-maximizing auction for a single good would allocate to a bidder with the highest virtual value, rather than a bidder with the highest value. To her, it seems like the revenue-maximizing auction should allocate to a bidder with the highest value, since such a bidder is willing to pay the most!

So Amy proposes the following revision to Myerson’s optimal auction: use the same personalized reserve prices as Myerson, so that bidders cannot win if they do not bid above their reserve, but allocate to a bidder with the highest value among those who bid above their reserve. More precisely, Amy’s allocation rule works as follows:

- Assume $v_i \sim F_i$ for all bidders $i \in [n]$, where $F_i$ is regular with bounded support.
- Determine the set $C$ of bidders $i$ who bid at least their reserve, namely $\phi_i^{-1}(0)$, where $\phi_i$ is bidder $i$’s virtual value function.
- Allocate the good to the highest bidder in $C$.

Amy also wants her auction to be IC and IR. Thus, she proposes using Myerson’s payment characterization lemma to set payments.

1. Explain Amy’s payment rule in words. What does the winning bidder pay?

2. When at most one bidder bids above their reserve prices, Amy’s and Myerson’s auctions yield the exact same outcome.

   Consider the case in which there are at least two bidders who bid above their reserve prices, and among them, the bidder with the highest value is not the same as the bidder with the highest virtual value. Prove that Amy’s auction yields weakly greater expected revenue than Myerson’s auction in this case.

3. Consider the other case, in which the bidder with the highest value also has the highest virtual value. In this case:
(a) Provide an example in which Amy’s auction produces greater revenue.

(b) Provide an example in which Myerson’s auction produces greater revenue.

Your examples should define the number of bidders, each bidder’s value distribution, and each bidder’s value.

4. **Extra Credit:** Suppose there are just two bidders: Bidder A whose value is drawn from $U(0,1)$ and Bidder B whose value is drawn from $U(0,2)$. Compute the expected revenue of both Amy’s and Myerson’s auctions.

4 A Simple(r) Approximately-Optimal Auction

Assuming regularity (i.e., weakly increasing virtual values), prove that the expected revenue of the second-price auction with personalized reserve prices is always at least half of the expected revenue of the optimal auction. In this simple(r) auction, the highest bidder wins as long as their bid exceeds their reserve, and they pay the maximum of the second-highest bid and their reserve.

**N.B.** This second-price auction with monopoly reserves is simpler than Myerson’s optimal auction, as it requires inverting each virtual value function only at 0, to compute reserve prices.

**Hint:** Here is an outline of how to proceed. Define OPT as the expected revenue of the optimal auction, and APX as the expected revenue of the simpler auction. The goal is to show $APX \geq \frac{1}{2} OPT$.

Assume bidder $i$ wins the optimal auction, and bidder $j$, the simple auction, so that $OPT = E_{v_i \sim F_i}[\phi_i(v_i)]$ and $APX = E_{v_j \sim F_j}[p_j]$. As the winners of the auctions, $i$ and $j$, are random variables,

$$OPT = E_{v_i \sim F_i}[\phi_i(v_i) | i = j] \Pr[i = j] + E_{v_i \sim F_i}[\phi_i(v_i) | i \neq j] \Pr[i \neq j]$$

the same winner different winners

Thus, it suffices to show (1) APX is at least the expected revenue of the optimal auction when $i = j$ (i.e., when the same bidder wins both auctions); and (2) APX is at least the expected revenue of the optimal auction when $i \neq j$ (i.e., when the winners of the two auctions differ): i.e.,

$$APX \geq E_{v_i \sim F_i}[\phi_i(v_i) | i = j] \Pr[i = j]$$

the same winner

$$APX \geq E_{v_i \sim F_i}[\phi_i(v_i) | i \neq j] \Pr[i \neq j],$$

different winners
from which it follows that $2\text{APX} \geq \text{OPT}$, because

$$
\text{OPT} = \mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i)] \\
= \mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \mid i = j] \Pr[i = j] \\
+ \mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \mid i \neq j] \Pr[i \neq j].
$$

the same winner
different winners