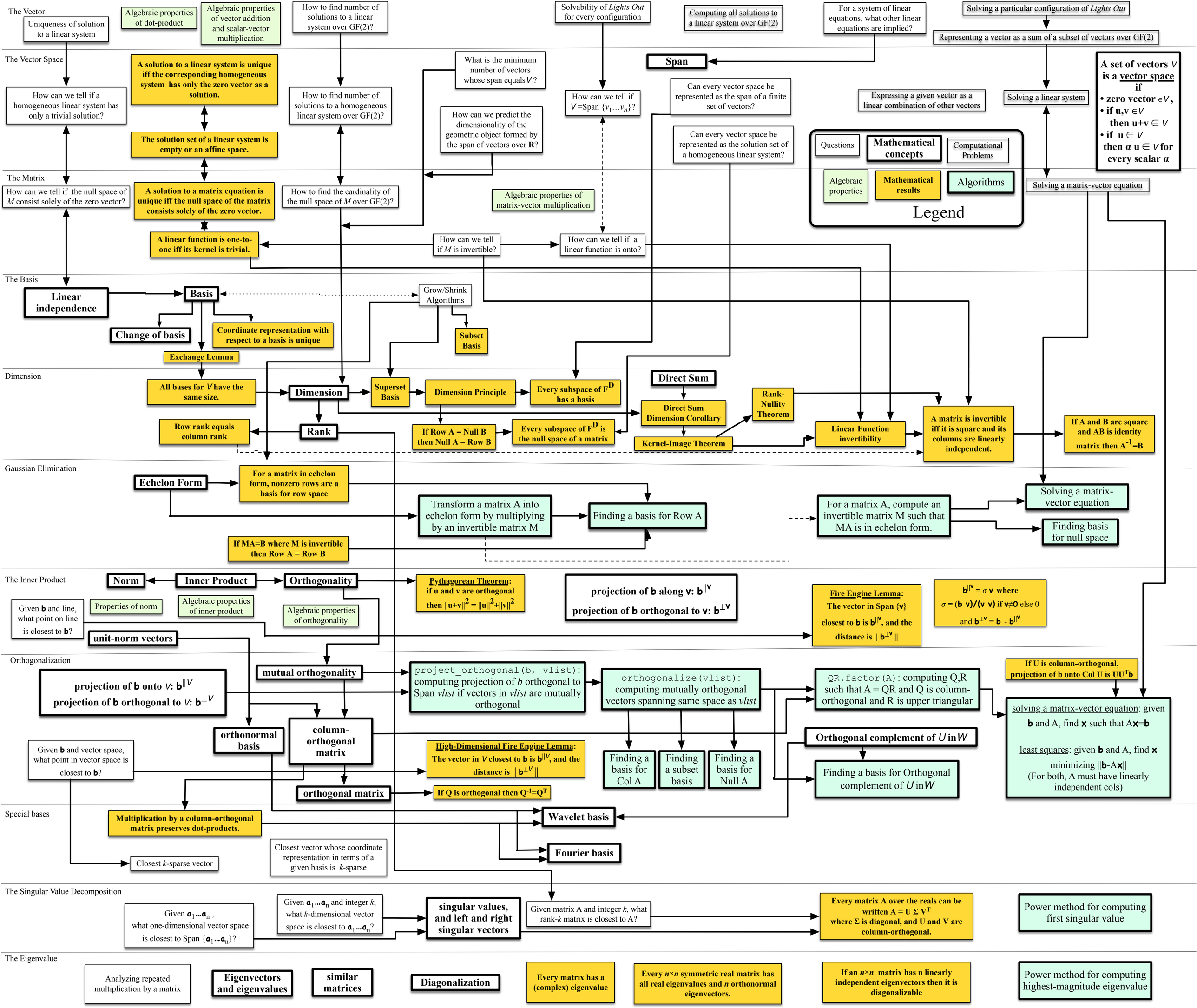
The Vector
Uniqueness of solution to a linear system
Algebraic properties of dot-product
Algebraic properties of vector addition and scalar-vector multiplication
How to find number of solutions to a linear system over GF(2)?
Solvability of Lights Out for every configuration
Computing all solutions to a linear system over GF(2)
For a system of linear equations, what other linear equations are implied?
Solving a particular configuration of Lights Out
Representing a vector as a sum of a subset of vectors over GF(2)
The Vector Space
A solution to a linear system is unique iff the corresponding homogeneous system has only the zero vector as a solution.
What is the minimum number of vectors whose span equals V?
Span
A set of vectors V is a vector space if
• zero vector ∈ V,
• if u,v ∈ V then u+v ∈ V
• if u ∈ V then α u ∈ V for every scalar α
How can we tell if a homogeneous linear system has only a trivial solution?
How to find number of solutions to a homogeneous linear system over GF(2)?
How can we tell if V=Span {v1…vn}?
Can every vector space be represented as the span of a finite set of vectors?
Expressing a given vector as a linear combination of other vectors
Solving a linear system
How can we predict the dimensionality of the geometric object formed by the span of vectors over R?
Can every vector space be represented as the solution set of a homogeneous linear system?
Questions
Mathematical concepts
Computational Problems
Algebraic properties
Mathematical results
Algorithms
Legend
The Matrix
The solution set of a linear system is empty or an affine space.
How can we tell if the null space of M consist solely of the zero vector?
A solution to a matrix equation is unique iff the null space of the matrix consists solely of the zero vector.
How to find the cardinality of the null space of M over GF(2)?
Algebraic properties of matrix-vector multiplication
Solving a matrix-vector equation
A linear function is one-to-one iff its kernel is trivial.
How can we tell if M is invertible?
How can we tell if a linear function is onto?
The Basis
Linear independence
Basis
Grow/Shrink Algorithms
Change of basis
Coordinate representation with respect to a basis is unique
Subset Basis
Exchange Lemma
Dimension
All bases for V have the same size.
Dimension
Superset Basis
Dimension Principle
Every subspace of F^D has a basis
Direct Sum
Direct Sum Dimension Corollary
Rank-Nullity Theorem
Row rank equals column rank
Rank
If Row A = Null B then Null A = Row B
Every subspace of F^D is the null space of a matrix
Kernel-Image Theorem
Linear Function invertibility
A matrix is invertible iff it is square and its columns are linearly independent.
If A and B are square and AB is identity matrix then A^-1=B
Gaussian Elimination
Echelon Form
For a matrix in echelon form, nonzero rows are a basis for row space
Transform a matrix A into echelon form by multiplying by an invertible matrix M
Finding a basis for Row A
For a matrix A, compute an invertible matrix M such that MA is in echelon form.
Solving a matrix-vector equation
Finding basis for null space
If MA=B where M is invertible then Row A = Row B
The Inner Product
Norm
Inner Product
Orthogonality
Pythagorean Theorem: if u and v are orthogonal then ||u+v||^2 = ||u||^2+||v||^2
projection of b along v: b^||v
projection of b orthogonal to v: b^⊥v
Fire Engine Lemma: The vector in Span {v} closest to b is b^||v, and the distance is || b^⊥v ||
b^||v = σ v where σ = (b v)/(v v) if v≠0 else 0 and b^⊥v = b - b^||v
Given b and line, what point on line is closest to b?
Properties of norm
Algebraic properties of inner product
Algebraic properties of orthogonality
unit-norm vectors
Orthogonalization
mutual orthogonality
projection of b onto V: b^||V
projection of b orthogonal to V: b^⊥V
project_orthogonal(b, vlist): computing projection of b orthogonal to Span vlist if vectors in vlist are mutually orthogonal
orthogonalize(vlist): computing mutually orthogonal vectors spanning same space as vlist
QR.factor(A): computing Q,R such that A = QR and Q is column-orthogonal and R is upper triangular
If U is column-orthogonal, projection of b onto Col U is UU^Tb
solving a matrix-vector equation: given b and A, find x such that Ax=b
least squares: given b and A, find x minimizing ||b-Ax||
(For both, A must have linearly independent cols)
orthonormal basis
column-orthogonal matrix
Orthogonal complement of U in W
Given b and vector space, what point in vector space is closest to b?
High-Dimensional Fire Engine Lemma: The vector in V closest to b is b^||V, and the distance is || b^⊥V ||
Finding a basis for Col A
Finding a subset basis
Finding a basis for Null A
Finding a basis for Orthogonal complement of U in W
orthogonal matrix
If Q is orthogonal then Q^-1=Q^T
Special bases
Multiplication by a column-orthogonal matrix preserves dot-products.
Wavelet basis
Fourier basis
Closest vector whose coordinate representation in terms of a given basis is k-sparse
Closest k-sparse vector
The Singular Value Decomposition
Given a1…an, what one-dimensional vector space is closest to Span {a1…an}?
Given a1…an and integer k, what k-dimensional vector space is closest to a1…an?
singular values, and left and right singular vectors
Given matrix A and integer k, what rank-k matrix is closest to A?
Every matrix A over the reals can be written A = U Σ V^T where Σ is diagonal, and U and V are column-orthogonal.
Power method for computing first singular value
The Eigenvalue
Analyzing repeated multiplication by a matrix
Eigenvectors and eigenvalues
similar matrices
Diagonalization
Every matrix has a (complex) eigenvalue
Every n×n symmetric real matrix has all real eigenvalues and n orthonormal eigenvectors.
If an n×n matrix has n linearly independent eigenvectors then it is diagonalizable
Power method for computing highest-magnitude eigenvalue