Intro to Counting

Michael L. Littman

CS 22 2020

March 9, 2020

Overview

Continuous vs. Discrete

continuous	discrete
sounds	

Continuous vs. Discrete

continuous	discrete
sounds	words
clay	

Continuous vs. Discrete

continuous	discrete
sounds	words
clay	legos
photographs	

Continuous vs. Discrete

continuous	discrete
sounds	words
clay	legos
photographs	diagrams
mouse	

Continuous vs. Discrete

continuous	discrete
sounds	words
clay	legos
photographs	diagrams
mouse	keyboard
sitting on the floor	

Continuous vs. Discrete

continuous	discrete
sounds	words
clay	legos
photographs	diagrams
mouse	keyboard
sitting on the floor	sitting in seats
$\mathbb{R}$	

Continuous vs. Discrete

continuous	discrete
sounds	words
clay	legos
photographs	diagrams
mouse	keyboard
sitting on the floor	sitting in seats
$\mathbb{R}$	$\mathbb{Z}$
quantities	

Continuous vs. Discrete

continuous	discrete
sounds	words
clay	legos
photographs	diagrams
mouse	keyboard
sitting on the floor	sitting in seats
$\mathbb{R}$	$\mathbb{Z}$
quantities	numbers
measuring	

Continuous vs. Discrete

continuous	discrete
sounds	words
clay	legos
photographs	diagrams
mouse	keyboard
sitting on the floor	sitting in seats
$\mathbb{R}$	$\mathbb{Z}$
quantities	numbers
measuring	**counting**

General strategy for counting

- Get good at counting some categories of things.

General strategy for counting

- Get good at counting some categories of things.
- Use bijections to relate one of those to the problem at hand.

General strategy for counting

- Get good at counting some categories of things.
- Use bijections to relate one of those to the problem at hand.
- The main category we will study is *counting sequences*.

Product rule

Rule: If $P_1, P_2, \ldots, P_n$ are finite sets, then:

$$|P_1 \times P_2 \times \cdots P_n| = |P_1| \cdot |P_2| \cdot \cdots \cdot |P_n|.$$

Product rule

Rule: If $P_1, P_2, \ldots, P_n$ are finite sets, then:

$$|P_1 \times P_2 \times \cdots P_n| = |P_1| \cdot |P_2| \cdot \cdots \cdot |P_n|.$$

Example:

- cakes = { chocolate, vanilla }

Product rule

Rule: If $P_1, P_2, \ldots, P_n$ are finite sets, then:

$$|P_1 \times P_2 \times \cdots P_n| = |P_1| \cdot |P_2| \cdot \cdots \cdot |P_n|.$$

Example:

- cakes = { chocolate, vanilla }
- toppings = { sprinkles, crumbs, none }
- fillings = { choc creme, white creme, jelly, custard, none }

If a donut consists of a choice of cake (one of 2), topping (one of 3), and filling (one of 5), how many different donuts are there?

Product rule

Rule: If $P_1, P_2, \ldots, P_n$ are finite sets, then:

$$|P_1 \times P_2 \times \cdots P_n| = |P_1| \cdot |P_2| \cdot \cdots \cdot |P_n|.$$

Example:

- cakes = { chocolate, vanilla }
- toppings = { sprinkles, crumbs, none }
- fillings = { choc creme, white creme, jelly, custard, none }

If a donut consists of a choice of cake (one of 2), topping (one of 3), and filling (one of 5), how many different donuts are there?
$2 \times 3 \times 5 = 30$.

Freestyle


ORIGINAL TASTE
Coca-Cola
Diet Coke
Coca-Cola zero SUGAR
CAFFEINE FREE Diet Coke
Diet Dr Pepper
Dr Pepper
Pibb Xtra
Pibb Zero
cherry
cherry vanilla
vanilla

Number of subsets rule

We showed that the number of subsets of a set of size n has a bijection with the set of binary strings of length n.

Number of subsets rule

We showed that the number of subsets of a set of size n has a bijection with the set of binary strings of length n.

Why are there 2^n binary strings of length n?

Number of subsets rule

We showed that the number of subsets of a set of size n has a bijection with the set of binary strings of length n.

Why are there 2^n binary strings of length n?

$$\{0,1\}^n = \{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}.$$

Number of subsets rule

We showed that the number of subsets of a set of size n has a bijection with the set of binary strings of length n.

Why are there 2^n binary strings of length n?

$$\{0,1\}^n = \{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}.$$

Product rule!

Number of subsets rule

We showed that the number of subsets of a set of size n has a bijection with the set of binary strings of length n.

Why are there 2^n binary strings of length n?

$$\{0,1\}^n = \{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}.$$

Product rule! $2 \cdot 2 \cdot \cdots \cdot 2 = 2^n$.

Sum rule

Rule: If $A_1, A_2, \ldots, A_n$ are disjoint sets, then:

$$|A1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Sum rule

Rule: If $A_1, A_2, \ldots, A_n$ are disjoint sets, then:

$$|A1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Example:

Sum rule

Rule: If $A_1, A_2, \ldots, A_n$ are disjoint sets, then:

$$|A1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Example:

- Mello Yello and Mello Yello Zero come in 8 flavors.

Sum rule

Rule: If $A_1, A_2, \ldots, A_n$ are disjoint sets, then:

$$|A1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Example:

- Mello Yello and Mello Yello Zero come in 8 flavors.
- Seagram's and Seagram's Diet come in 6 flavors.

Sum rule

Rule: If $A_1, A_2, \dots, A_n$ are disjoint sets, then:

$$|A1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Example:

- Mello Yello and Mello Yello Zero come in 8 flavors.
- Seagram's and Seagram's Diet come in 6 flavors.

Total?

Sum rule

Rule: If $A_1, A_2, \ldots, A_n$ are disjoint sets, then:

$$|A1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Example:

- Mello Yello and Mello Yello Zero come in 8 flavors.
- Seagram's and Seagram's Diet come in 6 flavors.

Total? $8 \cdot 2 + 6 \cdot 2 = 28$.

Counting Passwords (15.2.4)

Valid password:

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case).

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$
- 6-symbol password: letter symbol $\times$ 5 letter/digit symbols.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$
- 6-symbol password: letter symbol $\times$ 5 letter/digit symbols. $52 \cdot 62^5 = 5\text{e}{+}10$

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$
- 6-symbol password: letter symbol $\times$ 5 letter/digit symbols. $52 \cdot 62^5 = $ 5e+10
- 7-symbol password: letter symbol $\times$ 6 letter/digit symbols.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$
- 6-symbol password: letter symbol $\times$ 5 letter/digit symbols. $52 \cdot 62^5 =$ 5e+10
- 7-symbol password: letter symbol $\times$ 6 letter/digit symbols. $52 \cdot 62^6 =$3e+12

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$
- 6-symbol password: letter symbol $\times$ 5 letter/digit symbols. $52 \cdot 62^5 = 5e{+}10$
- 7-symbol password: letter symbol $\times$ 6 letter/digit symbols. $52 \cdot 62^6 = 3e{+}12$
- 8-symbol password: letter symbol $\times$ 7 letter/digit symbols.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$
- 6-symbol password: letter symbol $\times$ 5 letter/digit symbols. $52 \cdot 62^5 =$ 5e+10
- 7-symbol password: letter symbol $\times$ 6 letter/digit symbols. $52 \cdot 62^6 =$3e+12
- 8-symbol password: letter symbol $\times$ 7 letter/digit symbols. $52 \cdot 62^7 =$ 2e+14

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$
- 6-symbol password: letter symbol $\times$ 5 letter/digit symbols. $52 \cdot 62^5 =$ 5e+10
- 7-symbol password: letter symbol $\times$ 6 letter/digit symbols. $52 \cdot 62^6 =$3e+12
- 8-symbol password: letter symbol $\times$ 7 letter/digit symbols. $52 \cdot 62^7 =$ 2e+14
- valid password: 6-symbol password $\cup$ 7-symbol password $\cup$ 8-symbol password.

Counting Passwords (15.2.4)

Valid password: (1) Sequence of 6 to 8 symbols. (2) First symbol must be a letter (either case). (3) Rest are letters or digits.

- Digit symbol: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. 10
- Letter: { a, b, ..., z }. 26
- Letter symbol: letter $\times$ { upper, lower}. $26 \cdot 2 = 52$
- Letter/digit symbol: letter symbol $\cup$ digit symbol. $52 + 10 = 62$
- 6-symbol password: letter symbol $\times$ 5 letter/digit symbols. $52 \cdot 62^5 =$ 5e+10
- 7-symbol password: letter symbol $\times$ 6 letter/digit symbols. $52 \cdot 62^6 =$3e+12
- 8-symbol password: letter symbol $\times$ 7 letter/digit symbols. $52 \cdot 62^7 =$ 2e+14
- valid password: 6-symbol password $\cup$ 7-symbol password $\cup$ 8-symbol password. 5e+10 + 3e+12 + 2e+14 = 2e+14

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

There's a bijection between sequences and donut assignments.

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

There's a bijection between sequences and donut assignments.

If the five people are Julia, Julie, Michael, Thomas, and Tyler, I can give the first donut to Thomas, the second to Tyler, the third to Julie, the fourth to Julie.

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

There's a bijection between sequences and donut assignments.

If the five people are Julia, Julie, Michael, Thomas, and Tyler, I can give the first donut to Thomas, the second to Tyler, the third to Julie, the fourth to Julie. The sequence is: (Thomas, Tyler, Julie, Julie).

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

There's a bijection between sequences and donut assignments.

If the five people are Julia, Julie, Michael, Thomas, and Tyler, I can give the first donut to Thomas, the second to Tyler, the third to Julie, the fourth to Julie. The sequence is: (Thomas, Tyler, Julie, Julie). Every assignment of the 4 donuts to the people becomes a sequence.

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

There's a bijection between sequences and donut assignments.

If the five people are Julia, Julie, Michael, Thomas, and Tyler, I can give the first donut to Thomas, the second to Tyler, the third to Julie, the fourth to Julie. The sequence is: (Thomas, Tyler, Julie, Julie). Every assignment of the 4 donuts to the people becomes a sequence. Every sequence corresponds to a way to give the donuts to the people.

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

There's a bijection between sequences and donut assignments.

If the five people are Julia, Julie, Michael, Thomas, and Tyler, I can give the first donut to Thomas, the second to Tyler, the third to Julie, the fourth to Julie. The sequence is: (Thomas, Tyler, Julie, Julie). Every assignment of the 4 donuts to the people becomes a sequence. Every sequence corresponds to a way to give the donuts to the people. Bijection.

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

There's a bijection between sequences and donut assignments.

If the five people are Julia, Julie, Michael, Thomas, and Tyler, I can give the first donut to Thomas, the second to Tyler, the third to Julie, the fourth to Julie. The sequence is: (Thomas, Tyler, Julie, Julie). Every assignment of the 4 donuts to the people becomes a sequence. Every sequence corresponds to a way to give the donuts to the people. Bijection. There are $5^4 = 625$ sequences (product rule), so there are 625 ways of assigning the donuts to people.

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

There's a bijection between sequences and donut assignments.

If the five people are Julia, Julie, Michael, Thomas, and Tyler, I can give the first donut to Thomas, the second to Tyler, the third to Julie, the fourth to Julie. The sequence is: (Thomas, Tyler, Julie, Julie). Every assignment of the 4 donuts to the people becomes a sequence. Every sequence corresponds to a way to give the donuts to the people. Bijection. There are $5^4 = 625$ sequences (product rule), so there are 625 ways of assigning the donuts to people.

In general, n people, k donuts,

Counting assignments

I have 4 different donuts and 5 different people. How many different ways can I give the donuts to the people?

There's a bijection between sequences and donut assignments.

If the five people are Julia, Julie, Michael, Thomas, and Tyler, I can give the first donut to Thomas, the second to Tyler, the third to Julie, the fourth to Julie. The sequence is: (Thomas, Tyler, Julie, Julie). Every assignment of the 4 donuts to the people becomes a sequence. Every sequence corresponds to a way to give the donuts to the people. Bijection. There are $5^4 = 625$ sequences (product rule), so there are 625 ways of assigning the donuts to people.

In general, n people, k donuts, n^k assignments of the donuts to people.

Generalized product rule

If we add the constraint that no person gets more than one donut, the product rule no longer applies.

Generalized product rule

If we add the constraint that no person gets more than one donut, the product rule no longer applies. Specifically, the entries in the list now depend on each other and we can't count them by just multiplying.

Generalized product rule

If we add the constraint that no person gets more than one donut, the product rule no longer applies. Specifically, the entries in the list now depend on each other and we can't count them by just multiplying.

Rule: Let S be a set of length-k sequences. If there are:

Generalized product rule

If we add the constraint that no person gets more than one donut, the product rule no longer applies. Specifically, the entries in the list now depend on each other and we can't count them by just multiplying.

Rule: Let S be a set of length-k sequences. If there are:

- n_1 possible first entries,

Generalized product rule

If we add the constraint that no person gets more than one donut, the product rule no longer applies. Specifically, the entries in the list now depend on each other and we can't count them by just multiplying.

Rule: Let S be a set of length-k sequences. If there are:

- n_1 possible first entries,
- n_2 possible second entries for each first entry,

Generalized product rule

If we add the constraint that no person gets more than one donut, the product rule no longer applies. Specifically, the entries in the list now depend on each other and we can't count them by just multiplying.

Rule: Let S be a set of length-k sequences. If there are:

- n_1 possible first entries,
- n_2 possible second entries for each first entry,
- $\vdots$

Generalized product rule

If we add the constraint that no person gets more than one donut, the product rule no longer applies. Specifically, the entries in the list now depend on each other and we can't count them by just multiplying.

Rule: Let S be a set of length-k sequences. If there are:

- n_1 possible first entries,
- n_2 possible second entries for each first entry,
- $\vdots$
- n_k possible kth entries for each sequence of first $k-1$ entries,

then:

Generalized product rule

If we add the constraint that no person gets more than one donut, the product rule no longer applies. Specifically, the entries in the list now depend on each other and we can't count them by just multiplying.

Rule: Let S be a set of length-k sequences. If there are:

- n_1 possible first entries,
- n_2 possible second entries for each first entry,
- $\vdots$
- n_k possible kth entries for each sequence of first $k-1$ entries,

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_k.$$

Counting assignments with restrictions

With the 5 people and 4 donuts case, any of the 5 can be given the first donut.

Counting assignments with restrictions

With the 5 people and 4 donuts case, any of the 5 can be given the first donut. But, for the second donut, we cannot reuse whoever got the first one.

Counting assignments with restrictions

With the 5 people and 4 donuts case, any of the 5 can be given the first donut. But, for the second donut, we cannot reuse whoever got the first one. So, 4 choices left.

Counting assignments with restrictions

With the 5 people and 4 donuts case, any of the 5 can be given the first donut. But, for the second donut, we cannot reuse whoever got the first one. So, 4 choices left. For the third donut, there are only 3 choices of people left.

Counting assignments with restrictions

With the 5 people and 4 donuts case, any of the 5 can be given the first donut. But, for the second donut, we cannot reuse whoever got the first one. So, 4 choices left. For the third donut, there are only 3 choices of people left. And, for the fourth donut, there are only two people left.

Counting assignments with restrictions

With the 5 people and 4 donuts case, any of the 5 can be given the first donut. But, for the second donut, we cannot reuse whoever got the first one. So, 4 choices left. For the third donut, there are only 3 choices of people left. And, for the fourth donut, there are only two people left.

$5 \cdot 4 \cdot 3 \cdot 2 = 120$, which is less than the 625 unrestricted possibilities.

Counting assignments with restrictions

With the 5 people and 4 donuts case, any of the 5 can be given the first donut. But, for the second donut, we cannot reuse whoever got the first one. So, 4 choices left. For the third donut, there are only 3 choices of people left. And, for the fourth donut, there are only two people left.

$5 \cdot 4 \cdot 3 \cdot 2 = 120$, which is less than the 625 unrestricted possibilities. (How many ways are there to give out the donuts so that *someone* gets more than one donut?)

Counting assignments with restrictions

With the 5 people and 4 donuts case, any of the 5 can be given the first donut. But, for the second donut, we cannot reuse whoever got the first one. So, 4 choices left. For the third donut, there are only 3 choices of people left. And, for the fourth donut, there are only two people left.

$5 \cdot 4 \cdot 3 \cdot 2 = 120$, which is less than the 625 unrestricted possibilities. (How many ways are there to give out the donuts so that *someone* gets more than one donut?)

General form, n people, $k < n$ donuts, no one gets more than one:

Counting assignments with restrictions

With the 5 people and 4 donuts case, any of the 5 can be given the first donut. But, for the second donut, we cannot reuse whoever got the first one. So, 4 choices left. For the third donut, there are only 3 choices of people left. And, for the fourth donut, there are only two people left.

$5 \cdot 4 \cdot 3 \cdot 2 = 120$, which is less than the 625 unrestricted possibilities. (How many ways are there to give out the donuts so that *someone* gets more than one donut?)

General form, n people, $k < n$ donuts, no one gets more than one: $n \cdot (n-1) \cdot \cdots \cdot (n-k+1)$.

Serial numbers

Serial numbers

“Defective” because the serial number (11180915) repeats a digit (1).

Fraction non-defective

Assuming all 8-digit serial numbers equally likely, what fraction of bills are not defective?

Fraction non-defective

Assuming all 8-digit serial numbers equally likely, what fraction of bills are not defective?

How many 8-digit serial numbers?

Fraction non-defective

Assuming all 8-digit serial numbers equally likely, what fraction of bills are not defective?

How many 8-digit serial numbers? 10^8 by the product rule.

Fraction non-defective

Assuming all 8-digit serial numbers equally likely, what fraction of bills are not defective?

How many 8-digit serial numbers? 10^8 by the product rule.

How many 8-digit serials numbers have no repeats (not defective)?

Fraction non-defective

Assuming all 8-digit serial numbers equally likely, what fraction of bills are not defective?

How many 8-digit serial numbers? 10^8 by the product rule.

How many 8-digit serials numbers have no repeats (not defective)? $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 1814400$ by the generalized product rule.

Fraction non-defective

Assuming all 8-digit serial numbers equally likely, what fraction of bills are not defective?

How many 8-digit serial numbers? 10^8 by the product rule.

How many 8-digit serials numbers have no repeats (not defective)? $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 1814400$ by the generalized product rule.

Fraction not defective: not-defective / total = $1814400/100000000 = .018144$ or 1.8%.

Pawn, Bishop, Knight

How many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column?

Pawn, Bishop, Knight

How many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column?

There are 8 rows and 8 columns, which we'll number 1 to 8.

For example, P:(1,2), N:(4,1), B:(6,2) is invalid because the the pawn and bishop are in the samel column (2). Moving B to (6,3) fixes it.

Pawn, Bishop, Knight

How many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column?

There are 8 rows and 8 columns, which we'll number 1 to 8.

For example, P:(1,2), N:(4,1), B:(6,2) is invalid because the the pawn and bishop are in the samel column (2). Moving B to (6,3) fixes it.

P's row and column are unrestricted: $8 \cdot 8$. N only has 7 remaining choices of row and 7 remaining choices of column: $7 \cdot 7$. Finally, B has 6 choices for each of row and column.

Pawn, Bishop, Knight

How many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column?

There are 8 rows and 8 columns, which we'll number 1 to 8.

For example, P:(1,2), N:(4,1), B:(6,2) is invalid because the the pawn and bishop are in the samel column (2). Moving B to (6,3) fixes it.

P's row and column are unrestricted: $8 \cdot 8$. N only has 7 remaining choices of row and 7 remaining choices of column: $7 \cdot 7$. Finally, B has 6 choices for each of row and column.

So, $8 \cdot 8 \cdot 7 \cdot 7 \cdot 6 \cdot 6 = 112896$ different ways.

Definition

Definition: A *permutation* of a set is a sequence consisting of the elements of the set each repeated exactly once.

Definition

Definition: A *permutation* of a set is a sequence consisting of the elements of the set each repeated exactly once.

$S = \{w, x, y, z\}$

Definition

Definition: A *permutation* of a set is a sequence consisting of the elements of the set each repeated exactly once.

$S = \{w, x, y, z\}$

$(w, x, y, z), (w, x, z, y), (w, y, x, z), (w, y, z, x), (w, z, x, y), (w, z, y, x),$
$(x, w, y, z), (x, w, z, y), (x, y, w, z), (x, y, z, w), (x, z, w, y), (x, z, y, w),$
$(y, x, w, z), (y, x, z, w), (y, w, x, z), (y, w, z, x), (y, z, x, w), (y, z, w, x),$
$(z, x, y, w), (z, x, w, y), (z, y, x, w), (z, y, w, x), (z, w, x, y), (z, w, y, x),$

Definition

Definition: A *permutation* of a set is a sequence consisting of the elements of the set each repeated exactly once.

$S = \{w, x, y, z\}$

$(w, x, y, z), (w, x, z, y), (w, y, x, z), (w, y, z, x), (w, z, x, y), (w, z, y, x),$
$(x, w, y, z), (x, w, z, y), (x, y, w, z), (x, y, z, w), (x, z, w, y), (x, z, y, w),$
$(y, x, w, z), (y, x, z, w), (y, w, x, z), (y, w, z, x), (y, z, x, w), (y, z, w, x),$
$(z, x, y, w), (z, x, w, y), (z, y, x, w), (z, y, w, x), (z, w, x, y), (z, w, y, x),$

How many permutations on n elements?

Definition

Definition: A *permutation* of a set is a sequence consisting of the elements of the set each repeated exactly once.

$S = \{w, x, y, z\}$

$(w, x, y, z), (w, x, z, y), (w, y, x, z), (w, y, z, x), (w, z, x, y), (w, z, y, x),$
$(x, w, y, z), (x, w, z, y), (x, y, w, z), (x, y, z, w), (x, z, w, y), (x, z, y, w),$
$(y, x, w, z), (y, x, z, w), (y, w, x, z), (y, w, z, x), (y, z, x, w), (y, z, w, x),$
$(z, x, y, w), (z, x, w, y), (z, y, x, w), (z, y, w, x), (z, w, x, y), (z, w, y, x),$

How many permutations on n elements? It's like the non-repeating donut problem where $n = k$:

Definition

Definition: A *permutation* of a set is a sequence consisting of the elements of the set each repeated exactly once.

$S = \{w, x, y, z\}$

$(w, x, y, z), (w, x, z, y), (w, y, x, z), (w, y, z, x), (w, z, x, y), (w, z, y, x),$
$(x, w, y, z), (x, w, z, y), (x, y, w, z), (x, y, z, w), (x, z, w, y), (x, z, y, w),$
$(y, x, w, z), (y, x, z, w), (y, w, x, z), (y, w, z, x), (y, z, x, w), (y, z, w, x),$
$(z, x, y, w), (z, x, w, y), (z, y, x, w), (z, y, w, x), (z, w, x, y), (z, w, y, x),$

How many permutations on n elements? It's like the non-repeating donut problem where $n = k$:

$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1 = n!.$

Examples

- Assigning n professors n classes to teach (one each).

Examples

- Assigning n professors n classes to teach (one each).
- Assigning n actors to n roles (one each).

Examples

- Assigning n professors n classes to teach (one each).
- Assigning n actors to n roles (one each).
- Visit all of n cities in some order.

Examples

- Assigning n professors n classes to teach (one each).
- Assigning n actors to n roles (one each).
- Visit all of n cities in some order.
- Seating arrangements of n people.

Examples

- Assigning n professors n classes to teach (one each).
- Assigning n actors to n roles (one each).
- Visit all of n cities in some order.
- Seating arrangements of n people.
- Ways of shuffling n cards.