Binomial Theorem, Inclusion/Exclusion

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Overview

The Binomial Theorem (15.7)

Inclusion-Exclusion (15.12)
- Union of Two Sets (15.12.1)
- Union of Three Sets (15.12.2)
- Sequences with 42, 04, or 60 (15.12.3)
- Union of n Sets (15.12.4)
Historical perspective

I’m very well acquainted too with matters mathematical,
I understand equations, both the simple and quadratical,
About binomial theorem I am teeming with a lot o’ news—
With many cheerful facts about the square of the hypotenuse.

From “Modern Major General” by Arthur Sullivan and William Schwenck Gilbert (1879)
Binomials to powers: Examples

\[(a + b)^2 = aa + ab + ba + bb \]
\[= a^2 + 2ab + b^2 \]

\[(a + b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb \]
\[= a^3 + 3a^2b + 3ab^2 + b^3 \]

\[(a + b)^4 = aaaa + aaab + aaba + aabb + abaa + abab + abba + abbb + baaa + baab + baba + bbbb + bbaa + bbab + bbba + bbbb \]
\[= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \]

How about \((a + b)^n\)? How many terms consist of exactly \(k\) bs?
Since it’s all combinations of an \(a\) and \(b\) in each position, there are \(\binom{n}{k}\) such terms.
Binomial theorem

**Theorem:** For all $n \in \mathbb{N}$, $a, b \in \mathbb{R}$,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$ 

Sometimes $\binom{n}{k}$ called the *binomial coefficient* because of this connection.
Pets and sets

$S$: Set of all students in CS22.

$D \subseteq S$: Set of all students in CS22 who have a pet dog.

$C \subseteq S$: Set of all students in CS22 who have a pet cat.

$D \cup C$: Set of all students in CS22 who have a pet dog or cat.

$|D \cup C| = |D| + |C|$? Handles people who have neither correctly. Handles people who have one pet correctly. Messes up on people who have both.
Formulas for union

What's wrong with each formula for $|C \cup D|$?

- $|C| + |D|$? Double counted people who have both.
- $|C - D| + |D - C|$? Skipped people who have both.
- $|C - D| + |D - C| + |C \cap D|$? Actually, that should work. But, set difference can be tricky.
- $|C| + |D| - |C \cap D|$? Nailed it. Correct for double counting.
Inclusion-Exclusion rule for two sets

**Rule:** For two sets $S_1$ and $S_2$,

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|.$$ 

**Example:**

- $S_1 = \{\text{Tyler, Julie}\}$: HTAs with an $e$ in their name.
- $S_2 = \{\text{Julie, Julia}\}$: HTAs with an $j$ in their name.
- $S_1 \cap S_2 = \{\text{Julie}\}$: HTAs with both a $j$ and an $e$ in their name.
- $S_1 \cup S_2 = \{\text{Tyler, Julie, Julia}\}$: HTAs with either a $j$ or an $e$ in their name.
- $|\{\text{Tyler, Julie, Julia}\}| = |\{\text{Tyler, Julie}\}| + |\{\text{Julie, Julia}\}| - |\{\text{Julie}\}|$
Generalize to three sets

\[ S: \text{Set of all students in CS22.} \]
\[ D \subseteq S: \text{Set of all students in CS22 who have a pet dog.} \]
\[ C \subseteq S: \text{Set of all students in CS22 who have a pet cat.} \]
\[ B \subseteq S: \text{Set of all students in CS22 who have a pet bunny.} \]

How express |\(B \cup C \cup D\)| in terms of size of intersections of sets?
Visual analysis 1

\[ |B \cup C \cup D| = |B| + |C| + |D| + \ldots \]
Visual analysis 2

\[ |B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| \ldots \]
Visual analysis 3

\[ |B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| + |B \cap C \cap D| \]
Inclusion-Exclusion rule for three sets

**Rule:** For three sets $S_1$, $S_2$, $S_3$,

\[
|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| \\
- |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| \\
+ |S_1 \cap S_2 \cap S_3|.
\]

**Example:**

- $S_1 = \{ \text{Tyler, Julie} \}$: HTAs with an e in their name.
- $S_2 = \{ \text{Julie, Julia} \}$: HTAs with an j in their name.
- $S_3 = \{ \text{Tyler, Julie, Julia} \}$: HTAs with an l in their name.
- $S_1 \cap S_2 \cap S_3 = \{ \text{Julie} \}$: HTAs with both a j and an e and an l in their name.

\[
|\{ \text{Tyler, Julie, Julia} \}| = |\{ \text{Tyler, Julie} \}| + \\
|\{ \text{Julie, Julia} \}| + |\{ \text{Tyler, Julie, Julia} \}| - |\{ \text{Julie} \}| \\
- |\{ \text{Tyler, Julie} \}| - |\{ \text{Julie, Julia} \}| + |\{ \text{Julie} \}|.
\]
Sets of permutations

In how many permutations of the set \(\{0, 1, 2, \ldots, 9\}\) do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

Which of these permutations has this property?

- (4, 6, 5, 0, 1, 8, 3, 2, 9, 7) nope.
- (0, 4, 6, 1, 8, 5, 9, 3, 7, 2) 04!
- (3, 4, 2, 0, 5, 6, 1, 9, 8, 7) 42!
- (3, 9, 4, 1, 2, 7, 0, 5, 6, 8) nope.
- (0, 2, 6, 3, 7, 8, 4, 9, 5, 1) nope.

\[
\frac{2}{5} \times 10! = 1451520.
\]

\(P_{60}\): permutations of 0 through 9 that contain 60.

\(P_{04}\): permutations of 0 through 9 that contain 04.

\(P_{42}\): permutations of 0 through 9 that contain 42.

Want: \(|P_{60} \cup P_{04} \cup P_{42}|\).
Inclusion-exclusion, constrained permutation

\[
|P_{60} \cup P_{04} \cup P_{42}| \\
= |P_{60}| + |P_{04}| + |P_{42}| \\
- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
+ |P_{60} \cap P_{04} \cap P_{42}|
\]

\[|P_{60}| = ?\]

Clever trick: In \(P_{60}\), can view “60” as a unit. So, each element of \(P_{60}\) is a permutation of \(\{1, 2, 3, 4, 5, 7, 8, 9, 60\}\). Therefore, \(|P_{60}| = 9!\). \(|P_{04}| = 9!\). \(|P_{42}| = 9!\).
Pairwise intersections

\[ |P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! \]

\[ |P_{60} \cap P_{04}| =? \] Trick works again! Can view “604” as a unit. So, each element is a permutation of \{1, 2, 3, 5, 7, 8, 9, 604\}. Therefore, 8!.

\[ |P_{42} \cap P_{04}| =? \] Trick works again! Can view “042” as a unit. So, 8!.

\[ |P_{60} \cap P_{42}| =? \] Trick fails! Wait, no, just changes. Now, each element is a permutation of \{1, 3, 5, 7, 8, 9, 60, 42\}. Still 8!.
Three-way intersection

\[
|P_{60} \cup P_{04} \cup P_{42}| \\
= |P_{60}| + |P_{04}| + |P_{42}| \\
- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
+ |P_{60} \cap P_{04} \cap P_{42}| \\
= 3 \times 9! - 3 \times 8! + 7!
\]

\[
|P_{60} \cap P_{04} \cap P_{42}| = ? \text{. Yay, trick works again! Can view “6042” as a unit. So, each element is a permutation of } \{1, 3, 5, 7, 8, 9, 6042\}. \text{ Therefore, } 7!.
\]

\[
|P_{60} \cup P_{04} \cup P_{42}| = 3 \times 9! - 3 \times 8! + 7! = 972720.
\]

Sampling estimate was actually pretty good. Guessed 20% of permutations, but it’s closer to 27%.
n-way Inclusion-Exclusion

\[ |S_1 \cup S_2 \cup \cdots \cup S_n| = \]

the sum of the sizes of the individual sets
minus the sizes of all two-way intersections
plus the sizes of all three-way intersections
minus the sizes of all four-way intersections
plus the sizes of all five-way intersections, etc.

Hyper-mathy version:

\[ \left| \bigcup_{i=1}^{n} S_i \right| = \sum_{X \in \mathcal{P}([1,n])-\emptyset} (-1)^{|X|+1} \left| \bigcap_{i \in X} S_i \right| \]