Binomial Theorem, Inclusion/Exclusion

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Overview

The Binomial Theorem (15.7)

Inclusion-Exclusion (15.12)
  Union of Two Sets (15.12.1)
  Union of Three Sets (15.12.2)
    Sequences with 42, 04, or 60 (15.12.3)
  Union of n Sets (15.12.4)
Historical perspective

I’m very well acquainted too with matters mathematical,
I understand equations, both the simple and quadratical,
About binomial theorem I am teeming with a lot o’ news—
With many cheerful facts about the square of the hypotenuse.

From “Modern Major General” by Arthur Sullivan and William Schwenck Gilbert (1879)
Binomial Theorem, Inclusion/Exclusion

The Binomial Theorem (15.7)

Binomials to powers: Examples

\[(a + b)^2 = a^2 + 2ab + b^2\]

\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]

\[(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\]

How about \((a + b)^n\)? How many terms consist of exactly \(k\) bs?

Since it’s all combinations of an \(a\) and \(b\) in each position, there are \(\binom{n}{k}\) such terms.
Binomial theorem

**Theorem:** For all $n \in \mathbb{N}$, $a, b \in \mathbb{R}$,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Sometimes $\binom{n}{k}$ called the *binomial coefficient* because of this connection.
Pets and sets

* S: Set of all students in CS22.

* D ⊆ S: Set of all students in CS22 who have a pet dog.

* C ⊆ S: Set of all students in CS22 who have a pet cat.

* D ∪ C: Set of all students in CS22 who have a pet dog or cat.

|D ∪ C| = |D| + |C|? Handles people who have neither correctly. Handles people who have one pet correctly. Messes up on people who have both.
Formulas for union

What’s wrong with each formula for $|C \cup D|$?

- $|C| + |D|$? Double counted people who have both.
- $|C - D| + |D - C|$? Skipped people who have both.
- $|C - D| + |D - C| + |C \cap D|$? Actually, that should work. But, set difference can be tricky.
- $|C| + |D| - |C \cap D|$? Nailed it. Correct for double counting
Inclusion-Exclusion rule for two sets

**Rule**: For two sets \( S_1 \) and \( S_2 \),

\[ |S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|. \]

**Example:**

- \( S_1 = \{ \text{Tyler, Julie} \} \): HTAs with an \( e \) in their name.
- \( S_2 = \{ \text{Julie, Julia} \} \): HTAs with an \( j \) in their name.
- \( S_1 \cap S_2 = \{ \text{Julie} \} \): HTAs with both a \( j \) and an \( e \) in their name.
- \( S_1 \cup S_2 = \{ \text{Tyler, Julie, Julia} \} \): HTAs with either a \( j \) or an \( e \) in their name.
- \( |\{ \text{Tyler, Julie, Julia} \}| = |\{ \text{Tyler, Julie} \}| + |\{ \text{Julie, Julia} \}| - |\{ \text{Julie} \}| \)
Generalize to three sets

\( S \): Set of all students in CS22.
\( D \subseteq S \): Set of all students in CS22 who have a pet dog.
\( C \subseteq S \): Set of all students in CS22 who have a pet cat.
\( B \subseteq S \): Set of all students in CS22 who have a pet bunny.

How express \( |B \cup C \cup D| \) in terms of size of *intersections* of sets?
Visual analysis 1

\[ |B \cup C \cup D| = |B| + |C| + |D| + \ldots \]
Visual analysis 2

\[ |B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| \ldots \]
Visual analysis 3

\[ |B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| + |B \cap C \cap D| \]
Inclusion-Exclusion rule for three sets

**Rule:** For three sets $S_1$, $S_2$, $S_3$,

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|.$$

Example:

- $S_1 = \{ \text{Tyler, Julie} \}$: HTAs with an e in their name.
- $S_2 = \{ \text{Julie, Julia} \}$: HTAs with an j in their name.
- $S_3 = \{ \text{Tyler, Julie, Julia} \}$: HTAs with an l in their name.
- $S_1 \cap S_2 \cap S_3 = \{ \text{Julie} \}$: HTAs with both a j and an e and an l in their name.

$$|\{ \text{Tyler, Julie, Julia} \}| = |\{ \text{Tyler, Julie} \}| + |\{ \text{Julie, Julia} \}| + |\{ \text{Tyler, Julie, Julia} \}| - |\{ \text{Julie} \}| - |\{ \text{Tyler, Julie} \}| - |\{ \text{Julie, Julia} \}| + |\{ \text{Julie} \}|.$$
Sets of permutations

In how many permutations of the set \( \{0, 1, 2, \ldots, 9\} \) do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

Which of these permutations has this property?

- \( (4, 6, 5, 0, 1, 8, 3, 2, 9, 7) \) nope.
- \( (0, 4, 6, 1, 8, 5, 9, 3, 7, 2) \) 04!
- \( (3, 4, 2, 0, 5, 6, 1, 9, 8, 7) \) 42!
- \( (3, 9, 4, 1, 2, 7, 0, 5, 6, 8) \) nope.
- \( (0, 2, 6, 3, 7, 8, 4, 9, 5, 1) \) nope.

\[ \frac{2}{5} \times 10! = 1451520. \]

\( P_{60} \): permutations of 0 through 9 that contain 60.

\( P_{04} \): permutations of 0 through 9 that contain 04.

\( P_{42} \): permutations of 0 through 9 that contain 42.

Want: \( |P_{60} \cup P_{04} \cup P_{42}|. \)
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ |P_{60}| = ? \]

Clever trick: In \( P_{60} \), can view “60” as a unit. So, each element of \( P_{60} \) is a permutation of \{1, 2, 3, 4, 5, 7, 8, 9, 60\}. Therefore, \( |P_{60}| = 9! \). \( |P_{04}| = 9! \). \( |P_{42}| = 9! \).
Pairwise intersections

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! \]

\[ |P_{60} \cap P_{04}| = ? \text{ Trick works again! Can view “604” as a unit. So,} \]
\[ \text{each element is a permutation of } \{1, 2, 3, 5, 7, 8, 9, 604\}. \]
\[ \text{Therefore, } 8!. \]

\[ |P_{42} \cap P_{04}| = ? \text{ Trick works again! Can view “042” as a unit. So,} \]
\[ 8!. \]

\[ |P_{60} \cap P_{42}| = ? \text{ Trick fails! Wait, no, just changes. Now, each} \]
\[ \text{element is a permutation of } \{1, 3, 5, 7, 8, 9, 60, 42\}. \text{ Still } 8!. \]
Three-way intersection

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| 
+ |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! - 3 \times 8! 
+ |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60} \cap P_{04} \cap P_{42}| = ?. \] Yay, trick works again! Can view “6042” as a unit. So, each element is a permutation of \{1, 3, 5, 7, 8, 9, 6042\}. Therefore, 7!.

\[ |P_{60} \cup P_{04} \cup P_{42}| = 3 \times 9! - 3 \times 8! + 7! = 972720. \]

Sampling estimate was actually pretty good. Guessed 20% of permutations, but it’s closer to 27%.
n-way Inclusion-Exclusion

\[ |S_1 \cup S_2 \cup \cdots \cup S_n| = \]

the sum of the sizes of the individual sets
minus the sizes of all two-way intersections
plus the sizes of all three-way intersections
minus the sizes of all four-way intersections
plus the sizes of all five-way intersections, etc.

Hyper-mathy version:

\[ \left| \bigcup_{i=1}^{n} S_i \right| = \sum_{X \in \mathcal{P}([1,n])-\emptyset} (-1)^{|X|+1} \left| \bigcap_{i \in X} S_i \right| \]