# Site-Swap Notation

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# Overview

# Juggling demonstration

Examples:

- 2-ball shower.

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Example: 52512.

| toss | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 2 | 5 | 1 | 2 | 5 | 2 | 5 | 1 | 2 |

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Example: 52512.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 2 | 5 | 1 | 2 | 5 | 2 | 5 | 1 | 2 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

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Example: 52512.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 2 | 5 | 1 | 2 | 5 | 2 | 5 | 1 | 2 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

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Example: 52512.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 2 | 5 | 1 | 2 | 5 | 2 | 5 | 1 | 2 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 5 | 3 | 7 | 4 | 6 | 10 | 8 | 12 | 9 | 11 |

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| toss | 5 | 2 | 5 | 1 | 2 | 5 | 2 | 5 | 1 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
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| ball | A | B | C | B | B | A | B | C | B | B |

# Observations

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 2 | 5 | 1 | 2 | 5 | 2 | 5 | 1 | 2 | |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | |
| land | 5 | 3 | 7 | 4 | 6 | 10 | 8 | 12 | 9 | 11 | |
| land | 0 | 3 | 2 | 4 | 1 | 0 | 3 | 2 | 4 | 1 | (mod $n$) |
| ball | A | B | C | B | B | A | B | C | B | B | |

- $i$ is the beat, $n$ is the digits in the pattern.

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|---|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 2 | 5 | 1 | 2 | 5 | 2 | 5 | 1 | 2 | |
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- $t_i = t_{\text{rem}(i,n)}$.

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|---|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 2 | 5 | 1 | 2 | 5 | 2 | 5 | 1 | 2 | |
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| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | |
| land | 5 | 3 | 7 | 4 | 6 | 10 | 8 | 12 | 9 | 11 | |
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| toss | 5 | 2 | 5 | 1 | 2 | 5 | 2 | 5 | 1 | 2 | |
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| | | | | | | | | | | | |
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| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
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- $l_i = i + t_i$. Landings determined by tosses.
- For $i \in [0, n)$, $i \to \text{rem}(l_i, n)$ is a bijection.

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Case 2: $t_i$ is odd, so $t_i = 2k + 1$ for some integer $k$. The ball thrown at beat $i$ lands at $l_i = i + t_i = i + 2k + 1$.

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Case 1: $t_i$ is even, so $t_i = 2k$ for some integer $k$. The ball thrown at beat $i$ lands at $l_i = i + t_i = i + 2k$. Note that $i \equiv i + 2k \pmod 2$, so the ball will be thrown with the same hand the next time it is thrown.

Case 2: $t_i$ is odd, so $t_i = 2k + 1$ for some integer $k$. The ball thrown at beat $i$ lands at $l_i = i + t_i = i + 2k + 1$. Note that $i \not\equiv i + 2k + 1 \pmod 2$, so the ball will be thrown with the opposite hand the next time it is thrown. QED.

# Throw interpretations

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- 2: pause, hold the ball (or a tiny throw).

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- 0: empty hand. To clarify, the bijection mentioned earlier interprets 0 as ”no throw” and therefore there is no corresponding incoming ball (catch).
- 1: quick handoff to the other side.
- 2: pause, hold the ball (or a tiny throw).
- 3: “normal” throw from one hand to the other.

# Pattern 3

Consider the pattern 3.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| ball | A | B | C | A | B | C | A | B | C | A |

Ball A alternates hands, followed by B and C.

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| ball | A | B | C | A | B | C | A | B | C | A |

Ball A alternates hands, followed by B and C. Symmetric.

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Consider the pattern 3.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| ball | A | B | C | A | B | C | A | B | C | A |

Ball A alternates hands, followed by B and C. Symmetric. Known as the 3-ball cascade.

# Pattern 3

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| ball | A | B | C | A | B | C | A | B | C | A |

Ball A alternates hands, followed by B and C. Symmetric. Known as the 3-ball cascade. Makes a nice figure-8 pattern.

# Pattern 51

Contrast with pattern 3.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

# Pattern 51

Contrast with pattern 3.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 5 | 2 | 7 | 4 | 9 | 6 | 11 | 8 | 13 | 10 |

# Pattern 51

Contrast with pattern 3.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 5 | 2 | 7 | 4 | 9 | 6 | 11 | 8 | 13 | 10 |
| ball | A | B | B | C | C | A | A | B | B | C |

Hand 0 always throws a 5.

# Pattern 51

Contrast with pattern 3.

| toss | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 5 | 2 | 7 | 4 | 9 | 6 | 11 | 8 | 13 | 10 |
| ball | A | B | B | C | C | A | A | B | B | C |

Hand 0 always throws a 5. Hand 1 always throws a 1.

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 5 | 2 | 7 | 4 | 9 | 6 | 11 | 8 | 13 | 10 |
| ball | A | B | B | C | C | A | A | B | B | C |

Hand 0 always throws a 5. Hand 1 always throws a 1. Asymmetric.

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 |
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| land | 5 | 2 | 7 | 4 | 9 | 6 | 11 | 8 | 13 | 10 |
| ball | A | B | B | C | C | A | A | B | B | C |

Hand 0 always throws a 5. Hand 1 always throws a 1. Asymmetric. A makes a high circle from 0 to 1, then zips over to 0 and does it again.

# Pattern 51

Contrast with pattern 3.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 5 | 2 | 7 | 4 | 9 | 6 | 11 | 8 | 13 | 10 |
| ball | A | B | B | C | C | A | A | B | B | C |

Hand 0 always throws a 5. Hand 1 always throws a 1. Asymmetric. A makes a high circle from 0 to 1, then zips over to 0 and does it again. B and C follow suit.

# Pattern 51

Contrast with pattern 3.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 |
| beat | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| hand | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| land | 5 | 2 | 7 | 4 | 9 | 6 | 11 | 8 | 13 | 10 |
| ball | A | B | B | C | C | A | A | B | B | C |

Hand 0 always throws a 5. Hand 1 always throws a 1. Asymmetric. A makes a high circle from 0 to 1, then zips over to 0 and does it again. B and C follow suit. Known as the 3-ball shower.

# Pattern k

In general, what's the pattern with the form $k$?

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**Theorem**: The pattern $k$ is a symmetric $k$-ball pattern.

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Proof: For $k = 0$, empty hands.

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**Theorem**: The pattern $k$ is a symmetric $k$-ball pattern.

Proof: For $k = 0$, empty hands. For $k = 1$, pass the ball back and forth.

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**Theorem**: The pattern $k$ is a symmetric $k$-ball pattern.

Proof: For $k = 0$, empty hands. For $k = 1$, pass the ball back and forth. For $k = 2$, stand there holding the 2 objects.

# Pattern k

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**Theorem**: The pattern $k$ is a symmetric $k$-ball pattern.

Proof: For $k = 0$, empty hands. For $k = 1$, pass the ball back and forth. For $k = 2$, stand there holding the 2 objects. For $k \geq 3$, see above.

# Pattern 51 has the same number of objects as pattern 3

We already know they are both 3-ball patterns.

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We already know they are both 3-ball patterns. But, can we connect the two?

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| toss | 5 | 1 | 5 | 1 | 5 | 1 |
|---|---|---|---|---|---|---|

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We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |

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| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |

# Pattern 51 has the same number of objects as pattern 3

We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |

# Pattern 51 has the same number of objects as pattern 3

We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |

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| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |
| toss | 2 | $\underline{4}$ | $\overline{2}$ | 4 | 5 | 1 |

# Pattern 51 has the same number of objects as pattern 3

We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |
| toss | 2 | $\underline{4}$ | $\overline{2}$ | 4 | 5 | 1 |
| toss | 2 | $\underline{3}$ | $\overline{3}$ | 4 | 5 | 1 |

# Pattern 51 has the same number of objects as pattern 3

We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |
| toss | 2 | $\underline{4}$ | $\overline{2}$ | 4 | 5 | 1 |
| toss | 2 | $\underline{3}$ | $\overline{3}$ | 4 | 5 | 1 |
| toss | 2 | 3 | 3 | 4 | $\underline{5}$ | $\overline{1}$ |

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We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |
| toss | 2 | $\underline{4}$ | $\overline{2}$ | 4 | 5 | 1 |
| toss | 2 | $\underline{3}$ | $\overline{3}$ | 4 | 5 | 1 |
| toss | 2 | 3 | 3 | 4 | $\underline{5}$ | $\overline{1}$ |
| toss | 2 | 3 | 3 | 4 | $\underline{2}$ | $\overline{4}$ |

# Pattern 51 has the same number of objects as pattern 3

We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |
| toss | 2 | $\underline{4}$ | $\overline{2}$ | 4 | 5 | 1 |
| toss | 2 | $\underline{3}$ | $\overline{3}$ | 4 | 5 | 1 |
| toss | 2 | 3 | 3 | 4 | $\underline{5}$ | $\overline{1}$ |
| toss | 2 | 3 | 3 | 4 | $\underline{2}$ | $\overline{4}$ |
| toss | $\overline{2}$ | 3 | 3 | 4 | 2 | $\underline{4}$ |

# Pattern 51 has the same number of objects as pattern 3

We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |
| toss | 2 | $\underline{4}$ | $\overline{2}$ | 4 | 5 | 1 |
| toss | 2 | $\underline{3}$ | $\overline{3}$ | 4 | 5 | 1 |
| toss | 2 | 3 | 3 | 4 | $\underline{5}$ | $\overline{1}$ |
| toss | 2 | 3 | 3 | 4 | $\underline{2}$ | $\overline{4}$ |
| toss | $\overline{2}$ | 3 | 3 | 4 | 2 | $\underline{4}$ |
| toss | 3 | 3 | 3 | $\underline{4}$ | $\overline{2}$ | 3 |

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We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |
| toss | 2 | $\underline{4}$ | $\overline{2}$ | 4 | 5 | 1 |
| toss | 2 | $\underline{3}$ | $\overline{3}$ | 4 | 5 | 1 |
| toss | 2 | 3 | 3 | 4 | $\underline{5}$ | $\overline{1}$ |
| toss | 2 | 3 | 3 | 4 | $\underline{2}$ | $\overline{4}$ |
| toss | $\overline{2}$ | 3 | 3 | 4 | 2 | $\underline{4}$ |
| toss | 3 | 3 | 3 | $\underline{4}$ | $\overline{2}$ | 3 |
| toss | 3 | 3 | 3 | $\underline{3}$ | $\overline{3}$ | 3 |

# Pattern 51 has the same number of objects as pattern 3

We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |
| toss | 2 | $\underline{4}$ | $\overline{2}$ | 4 | 5 | 1 |
| toss | 2 | $\underline{3}$ | $\overline{3}$ | 4 | 5 | 1 |
| toss | 2 | 3 | 3 | 4 | $\underline{5}$ | $\overline{1}$ |
| toss | 2 | 3 | 3 | 4 | $\underline{2}$ | $\overline{4}$ |
| toss | $\overline{2}$ | 3 | 3 | 4 | 2 | $\underline{4}$ |
| toss | 3 | 3 | 3 | $\underline{4}$ | $\overline{2}$ | 3 |
| toss | 3 | 3 | 3 | $\underline{3}$ | $\overline{3}$ | 3 |

From 51 to 3 keeping numbers of balls constant.

# Pattern 51 has the same number of objects as pattern 3

We already know they are both 3-ball patterns. But, can we connect the two?

| | | | | | | |
|---|---|---|---|---|---|---|
| toss | 5 | 1 | 5 | 1 | 5 | 1 |
| toss | $\underline{5}$ | $\overline{1}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | $\underline{2}$ | $\overline{4}$ | $\overline{5}$ | 1 | 5 | $\underline{1}$ |
| toss | 2 | 4 | $\underline{5}$ | $\overline{1}$ | 5 | 1 |
| toss | 2 | 4 | $\underline{2}$ | $\overline{4}$ | 5 | 1 |
| toss | 2 | $\underline{4}$ | $\overline{2}$ | 4 | 5 | 1 |
| toss | 2 | $\underline{3}$ | $\overline{3}$ | 4 | 5 | 1 |
| toss | 2 | 3 | 3 | 4 | $\underline{5}$ | $\overline{1}$ |
| toss | 2 | 3 | 3 | 4 | $\underline{2}$ | $\overline{4}$ |
| toss | $\overline{2}$ | 3 | 3 | 4 | 2 | $\underline{4}$ |
| toss | 3 | 3 | 3 | $\underline{4}$ | $\overline{2}$ | 3 |
| toss | 3 | 3 | 3 | $\underline{3}$ | $\overline{3}$ | 3 |

From 51 to 3 keeping numbers of balls constant. So, both 3.

# Averaging theorem

**Theorem**: The average of the digits in a pattern equals the number of balls in the pattern.

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Proof sketch: Can use an induction argument to show we can reduce any pattern to a $k$ pattern using *site swaps*.

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**Theorem**: The average of the digits in a pattern equals the number of balls in the pattern.

Proof sketch: Can use an induction argument to show we can reduce any pattern to a $k$ pattern using *site swaps*. A site swap doesn't change the average because replaces $a$ and $b$ with $b-1$ and $a+1$.