Division Rule, Counting Subsets

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Overview

Division rule

Definition: A k-to-1 function maps exactly k elements of the domain to every element of the codomain.

In the "4-legged zoo" song, the function mapping each leg to its owner is 4-to-1.

- A: legs
- B: animals
- f: leg $a \in A$ belongs to animal $b \in B$
- Every element in B is pointed to by 4 elements in A

Rule: If $f : A \to B$ is k-to-1, then $|A| = k|B|$.

Example: Highway counter counts axles. At the end of the day, it counted 2742 axles. How many buses? Mapping from axles to buses is 2-to-1, so axles is twice the buses. So, $2742/2 = 1371$ buses.

Rook placements

We counted placements of a pawn, knight, and bishop, so no two occupied the same row or column via the generalized product rule.

What about two identical rooks?

- A: all sequences (r_1, c_1, r_2, c_2) where r_1 and r_2 are distinct rows and c_1 and c_2 are distinct columns.
- B: all valid rook configurations.
- f: map the sequence (r_1, c_1, r_2, c_2) to a configuration with one rook at row r_1, column c_1 and the other at row r_2, column c_2.

Note: $(6, 2, 4, 1)$ and $(4, 1, 6, 2)$ are actually the same because the rooks are *identical*.

Double counted

Our function f maps *exactly* two sequences to every board configuration, one assigning the rooks in either order.

Thus, f is a 2-to-1 function. By the division rule, $|A| = 2|B|$. By the generalized product rule, $|A| = 8^2 \cdot 7^2 = 3136$. Since we double counted, the number of configurations is half that, 1568.

Circular ordering

If we're going to seat 4 people around a table, we can do it in $4! = 24$ ways. For example, dinner with my spouse and kids could be: (Max, Michael, Lisa, Molly) or (Lisa, Molly, Max, Michael).

Note, however, these two orderings induce the same ordered-next-to-relations. Michael has Max to his left and Lisa to his right in both.

If we only care about ordered-next-to-relations, how many orderings are there?

Formal seating

- A: A *seating* is determined by the sequence of diners going clockwise around the table starting at the top seat. Seatings are formally equivalent permutations of the n diners.
- B: An *arrangement* is determined by the sequence of diners going clockwise around the table starting after diner number 1, so it is formally the same as the permutations of diners 2 through n.
- f: We can map each permutation $a \in A$ to an arrangement $b \in B$ by seating the first diner in the permutation at the top of the table, putting the second diner to his left, the third diner to the left of the second, and so forth all the way around the table.

This mapping is n-to-1, since all n cyclic shifts of any seating a maps to the same arrangement b.

Seatings to arrangements

number of arrangements	
$= \lvert B \rvert$	defn of B
$= \lvert A \rvert / n$	division rule
$= n!/n$	generalized product rule
$= (n-1)!$	factorial defn

I also think of it as not having a choice where the first person sits (because that decision establishes anchor of the arrangement).

Choose

How many subsets of an n-element set? 2^n.

What if we insist that the subsets have k elements?

- Room in luggage for 4 of my 20 shirts.
- Deal a 5-card hand from 52 cards.
- Sundae with 3 ice cream choices from 32 flavors.

$$\binom{n}{k}$$

The number of k-element subsets of an n-item set. "n choose k".

Answers: $\binom{20}{4}$, $\binom{52}{5}$, $\binom{32}{3}$.

Special cases

3 laser pointers, choose one for lecture. How many possibilities? 3.

In general, n items choose 1? $\binom{n}{1} = n$.

3 HTAs, choose 3 for a staff meeting. How many possibilities? 1.

In general, n items choose n? $\binom{n}{n} = 1$.

Deriving the subset rule

- A: all permutations of n elements.
- B: separating n elements into k chosen and $n-k$ not chosen.
- f: Take permutation $a \in A$ and map it to $b \in B$ such that the first k elements in a are the chosen elements in b.

The function f is $k!(n-k)!$-to-1 because every rearrangement of the first k and every rearrangement of the last $n-k$ go to the same $b \in B$.

Subset rule

By the division rule: $n! = k!(n-k)!\binom{n}{k}$.

Rule:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

An identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof: Algebraically, tricky. But, can reason it out in terms of the definitions!

$\binom{n}{k}$: The number of different ways we can choose k items out of a set of n.

Induction-style, we can choose our k items all from the first $n-1$ *or* we can choose the last item and $k-1$ items from the first $n-1$.

That's, $\binom{n-1}{k} + \binom{n-1}{k-1}$. QED.

$[\underbrace{o, o, o, o, \ldots, o, o, o}_{n}]$ vs. $[\underbrace{o, o, o, o, \ldots, o, o}_{n-1}, o]$

k bits out of n

How many n-bit sequences contain exactly k ones? We talked about the bijection between subsets of an n-element set and n-bit sequences.

Example: A 4-element subset of $\{x_1, x_2, \ldots, x_8\}$ and the associated 8-bit sequence:

$$\begin{array}{cccccccccc} \{ & & x_2, & & x_4, & & & x_7, & x_8 & \} \\ (& 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 &). \end{array}$$

By the subset rule, the number of n-bit sequences with exactly k 1s is $\binom{n}{k}$.

Flashback

Remember this slide from last month?

Time to make the donuts

$A =$ all ways to select a half-dozen doughnuts when three varieties are available

$B =$ all 8-bit sequences with exactly two 1s

$$\underbrace{0\,0\,0\,0}_{\text{Boston cream}} \quad \underbrace{\qquad}_{\text{coconut}} \quad \underbrace{0\,0}_{\text{glazed}}$$

Put our 6 donuts into the three bins. Note: Every way we can choose donuts becomes a pattern. And, every pattern (with 6 donuts) corresponds to a valid choice.

Use 1s to mark the gaps.

$\underbrace{0\,0\,0\,0}_{\text{Boston cream}} \quad 1 \quad \underbrace{\qquad}_{\text{coconut}} \quad 1 \quad \underbrace{0\,0}_{\text{glazed}}$ Now, squeeze: 00001100.

Every 8-bit pattern with exactly 2 ones is a valid donut order. Every valid donut order can be encoded with an 8-bit pattern with exactly 2 ones. Same size!

Time to solve the donuts!

$A =$ all ways to select a half-dozen doughnuts when three varieties are available

$B =$ all 8-bit sequences with exactly two 1s

Now, we know that $|B| = \binom{8}{2} = 28$, so 28 ways to select a half-dozen donuts when three varieties available! Closure!

Donuts and separators rule

Rule: All ways to select n donuts when k varieties are available:

$$\binom{n+k-1}{k-1}$$

There's $k-1$ separators (1 bits) and n donuts (0 bits). Need to put the $k-1$ separators into the $n+k-1$ slots.