Induction

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Overview

Counting things

We went to the four-legged zoo
To visit our four-footed friends
Lions and tigers, cats and dogs
A goat and a cow and a couple o' hogs

Well, Miss Simpson said
If we counted every head on these quadrupeds
Then multiplied that number by four
We'd know how many feet went through the door
If we turned them all loose

"The Four-Legged Zoo" by Bob Dorough.

You can count things by counting things they are related to. In this case, heads and legs.

Counting sets

Take a set $A = \{a_1, a_2, \ldots, a_n\}$. How big is the powerset of A?

We can associate each subset with a binary number of length n. The first bit represents whether a_1 is in the subset. It can be either in (1) or out (0). The second represents whether a_2 is in the subset. It, too, can be either in (1) or out (0). Continue on up to a_n.

We have a bijection between n-bit strings and subsets of A. That's because every bit string encodes a unique subset and every subset is encoded by a unique string.

That means they are the same size. And, the number of n-bit strings is 2^n because each one encodes a counting number from 0 to $2^n - 1$. So, 2^n must also be the number of subsets.

Time to make the donuts

$A =$ all ways to select a half-dozen doughnuts when three varieties are available

$B =$ all 8-bit sequences with exactly two 1s

$$\underbrace{0\,0\,0\,0}_{\text{Boston cream}} \quad \underbrace{\qquad}_{\text{coconut}} \quad \underbrace{0\,0}_{\text{glazed}}$$

Put our 6 donuts into the three bins. Note: Every way we can choose donuts becomes a pattern. And, every pattern (with 6 donuts) corresponds to a valid choice.

Use 1s to mark the gaps.

$$\underbrace{0\,0\,0\,0}_{\text{Boston cream}} \quad 1 \quad \underbrace{\qquad}_{\text{coconut}} \quad 1 \quad \underbrace{0\,0}_{\text{glazed}}$$ Now, squeeze: 00001100.

Every 8-bit pattern with exactly 2 ones is a valid donut order. Every valid donut order can be encoded with an 8-bit pattern with exactly 2 ones. Same size!

Donut distribution

I have thirty boxes of a dozen donuts and a class of 320 students. Assume everyone wants a donut. I have everyone in the class form a single line. I give out donuts according to the following rule:

1. The zero-th person in line gets a donut.
2. I give a donut to anyone if the person directly ahead of them in line has gotten a donut.

Who gets a donut?

Everyone. Why? Because the donuts propagate down the line until everyone has one.

What if we only included the first rule? Only the zero-th person is guaranteed to get a donut.

What if we only included the second rule? No one would be guaranteed to get a donut.

Infinite donuts

What if my bag of donuts and my line of students was *infinite* and we followed the same two rules?

Would the zero-th person get a donut? Yes, rule one.

Would the first person get a donut? Yes, rule one and rule two.

Would the second person get a donut? Yes, rule one and rule two twice.

Would the $n \in \mathbb{N}$ person get a donut? Yes, rule one and rule two n times.

Would the ∞ person get a donut? No, ∞ is not a number. But, everyone else should be happy.

Even though the *process* never ends, we can say that everyone gets a donut. (The students deep in the line might have to wait, but they *will* get one.)

The Principle of Induction

Let P be a predicate on nonnegative integers. If

- $P(0)$ is true, and
- $P(n)$ IMPLIES $P(n+1)$ for all nonnegative integers, n,

then

- $P(m)$ is true for all nonnegative integers, m.

It's a great way to prove a *global* property $(\forall n \in \mathbb{N}, P(n))$ from two *local* properties ($P(0)$, $P(n)$ implies $P(n+1)$).

Bowling pins

Theorem: For all $n \in \mathbb{N}$,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Proof: Define $P(n)$ to be the claim that the sum of the numbers from 1 to n is $n(n+1)/2$. We want to show that $P(n)$ is true for all $n \in \mathbb{N}$.

$P(0)$ is true because the sum of no numbers is zero and $0 \cdot 1/2$ is also zero.

Bowling pins, Induction Step

Now, we need to show that IF $P(n)$ is true, then $P(n+1)$ must be true. So, let's assume $P(n)$ is true. That means the sum of the numbers from 1 to n is $n(n+1)/2$ for that n. (We're not assuming it's true for all n.)

$$
\begin{aligned}
& 1+2+3+\ldots+(n+1) \\
& \quad = \quad 1+2+3+\ldots+n+(n+1) \\
& \quad = \quad n(n+1)/2+(n+1) \\
& \quad = \quad (n+1)(n/2+1) \\
& \quad = \quad (n+1)(n+2)/2
\end{aligned}
$$

That means $P(n+1)$ is true. Since we have $P(0)$ and also $P(n)$ implies $P(n+1)$, we have $\forall m \in \mathbb{N}, P(m)$ by induction. QED.

Bowling pins, Observations

- What we showed is that the amount that $n(n+1)/2$ increases when n increases by 1 is exactly $n+1$.
- You can think of it as a summary of checking $P(0)$, $P(1)$, $P(2)$, $P(3)$ and then getting bored and giving a recipe that lets us check any other P we want on the fly.
- My internal sense is that I process induction with the same part of my brain that I use to process recursion. It requires a little mental adjusting before it feels natural, but you can get there.

A template

1. State that the proof uses induction.
2. Make the appropriate induction hypothesis $P(n)$ explicit.
3. Prove that the base case $P(0)$ is true.
4. Prove the induction step: $P(n)$ implies $P(n+1)$ for every nonnegative integer n.
5. Invoke induction and conclude that $P(n)$ is true for all nonnegative n.

Once more, with less feeling

Proof: We use induction. The induction hypothesis, $P(n)$, is

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \tag{1}$$

Base case: $P(0)$ is true, because both sides of Equation 1 equal zero when $n = 0$.

Inductive step: Assume that $P(n)$ is true, where n is any nonnegative integer. Then,

$$\begin{aligned} &1 + 2 + 3 + \ldots + n + (n+1) \\ &\quad = \tfrac{n(n+1)}{2} + (n+1) \quad \text{(by induction hypothesis)} \\ &\quad = \tfrac{(n+1)(n+2)}{2}, \quad \text{(by simple algebra)} \end{aligned}$$

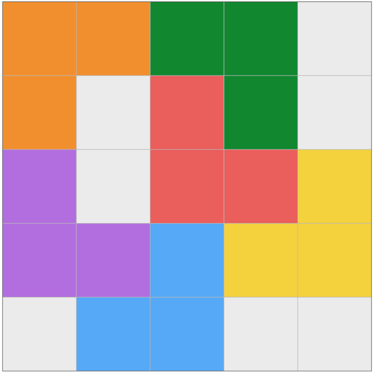
which proves $P(n+1)$.

So, it follows by induction that $P(n)$ is true for all nonnegative n. QED.

Le Definitions

- A *space* is a square.
- A *board* is an $n \times m$ rectangle of spaces with adjacent spaces touching.
- A *tile* is a set of touching squares that can go on top of an identically-shaped set of spaces.
- An *L-shaped tile* is 3 squares where two are touching a central third on consecutive sides.
- A *center space* of a square $n \times n$ board is any of spaces $(n/2-1, n/2-1)$, $(n/2+1, n/2-1)$, $(n/2-1, n/2+1)$, or $(n/2+1, n/2+1)$ if n is even.
- A *placement* of tiles on a board assigns each square in a tile to each space on the board so adjacency relationships are maintained and no squares are assigned to the same space. (Injection!)
- A *hole* is a cell not assigned a tile in a given placement.

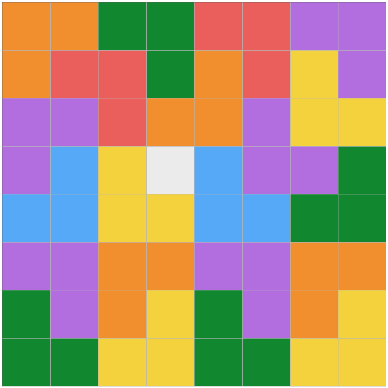
Le Picture



space? board? tile? L-shared tile? center space? placement? hole?

Le Theorem

Theorem: For all $n \geq 0$, there exists a tiling of a $2^n \times 2^n$ board using L-shaped tiles such that there is exactly one hole and it is on a center space.

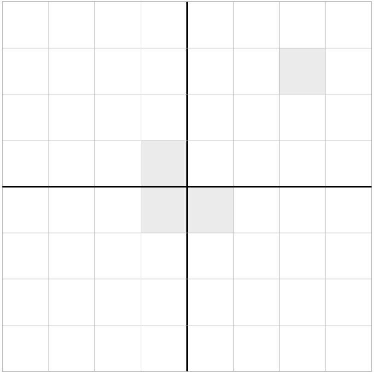
Example: 

Le Base Case

Proof: The proof is by induction. Let $P(n)$ be the proposition that, for *any* selected space, there exists a tiling of a $2^n \times 2^n$ board using L-shaped tiles such that there is exactly one hole and it is on the selected space.

Base case: $P(0)$ is true because there is only one space and there is a tiling (place no tiles) that has exactly one hole and it is in that space.

Le Inductive Step: Picture



Le Inductive Step: Words

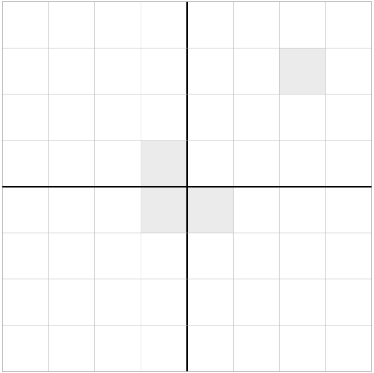
Assume that $P(n)$ is true for some $n \geq 0$; that is, for every space on a $2^n \times 2^n$ board, there exists a placement of L-shaped tiles that leaves only the chosen space empty.

Divide the $2^{n+1} \times 2^{n+1}$ board into four quadrants, each $2^n \times 2^n$. One quadrant contains the cell we want to leave empty. Select empty cells for the other three quadrants that are their corners that are center cells for the full board. By four applications of our inductive hypothesis, we can make a placement of L-shaped tiles that leave only those empty cells. Now, add one more L-shaped tile to the board, covering the undesired center cells.

This argument proves that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. Thus, $P(m)$ is true for all $m \in \mathbb{N}$, and the theorem follows as a special case where we choose the empty space to be a center space.

Le Theorem: Repeated

Theorem: For all $n \geq 0$, there exists a tiling of a $2^n \times 2^n$ board using L-shaped tiles such that there is exactly one hole and it is on a center space.



Challenge: Write a recursive *algorithm* that solves the problem.

When proofs go bad

There's a number of ways an induction proof can go off the rails.

- Didn't prove base case. Every number n equals $n+1$. If $P(n)$ is true, $n = n+1$. Adding one to both sides shows $P(n+1)$, $n+1 = n+2$. But, $P(0)$ is not true!
- Didn't really prove the inductive step. There's always room for more students in CS22. There's room for zero. If we can fit n people, we can always squeeze in one more. Not really.
- Proved the base case and inductive step, but not for all $n \geq 0$. All donuts are the same flavor. True of 1 donut. If it's true of n donuts, then if you give me $n+1$ donuts, the first n are the same flavor and the last n are the same flavor. Both are the same flavor as any of the middle donuts. Doesn't hold for $n = 1$.