Equivalence Relation and Bijection Examples

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CS 22 2020

February 5, 2020

Overview

Functions

Let R be a relation with domain A and codomain B.

If

$\forall a \in A, \forall b_1 \in B, \forall b_2 \in B, a\ R\ b_1$ and $a\ R\ b_2$ IMPLIES $b_1 = b_2$,

then R is a partial function.

If it's also the case that

$\forall a \in A, \exists b \in B, a\ R\ b$,

then R is a function.

We often write functions as $f(a)$, where $f : A \to B$, meaning we give f an $a \in A$ and it returns the $b \in B$ such that $a\ f\ b$.

Function image

Confusingly, we write $f(X)$ to suggest a function that takes a set as input and outputs the image of the set X under relation f. It depends whether X is an element of the domain or a subset of the domain (an element of the powerset of the domain).

If f is a function with domain A and codomain B, $f(A)$ is the *range* of the function.

Often $f(A) = B$, but not necessarily!

Function Composition

If $f : A \to B$ and $g : B \to C$ are functions, then we can *compose* them.

For any $a \in A$, $g(f(a)) \in C$.

Sometimes, we write $(g \circ f)(a)$.

If f is a function on A, we can write $f \circ f = f^2$. And, $f \circ f \circ f = f^3$. And, $f^m \circ f^n = f^{m+n}$.

So, what should f^1 be? f^0? f^{-1}?
$f^1 = f$, f^0 is the identity function $f^0(a) = a$, and f^{-1} is the inverse. Which it already is!

More relation properties

Let R be a binary relation on A.

Definition: R is *transitive* iff $\forall a, b, c \in A, a \ R \ b$ AND $b \ R \ c$ IMPLIES $a \ R \ c$.

Definition: R is *reflexive* iff $\forall a \in A, a \ R \ a$.

Definition: R is *irreflexive* iff $\forall a \in A, a \not R \ a$.

Definition: R is *symmetric* iff $\forall a, b \in A, a \ R \ b$ IMPLIES $b \ R \ a$.

Definition: R is *anti-symmetric* iff $\forall a \neq b \in A, a \ R \ b$ IMPLIES $b \not R \ a$.

RST Examples

In a group of people:

- reflexive and symmetric but not transitive: "is standing no more than 4 feet away from"
- reflexive and transitive but not symmetric: "is no taller than"
- transitive and symmetric but not reflexive: "is in CS22 with" (assuming some people are not in CS22).

All of the above

Definition: A relation that is reflexive, symmetric, and transitive is an *equivalence* relation.

The relation *partitions* the domain.

Example: "rounds to the same value as".

- Reflexive: x rounds to the same value as x.
- Symmetric: If x rounds to the same value as y, then y rounds to the same value as x.
- Transitive: If x rounds to the same value as y, and y rounds to the same value as z, then x must round to the same value as z.

We're really just restating key properties of the equality relation.

Equivalence classes

If R is an equivalence relation on A, we can partition the elements of A into sets $[a]_R = \{a' \in A | a \; R \; a'\}$.

Here, the a in $[a]_R$ is an arbitrarily chosen representitive of its *equivalence class*.

Example: Define relation R on $\mathbb{Z}$ by $x \; R \; y$ iff x mod $3 = y$ mod 3.

Then, we have:

- $[0]_R = \{\ldots, -3, 0, 3, 6, \ldots\}$
- $[1]_R = \{\ldots, -2, 1, 4, 7, \ldots\}$
- $[2]_R = \{\ldots, -1, 2, 5, 8, \ldots\}$

These three sets are a *partition* of the integers. That means mutually exclusive and exhaustive. That means $[0]_R \cup [1]_R \cup [2]_R = \mathbb{Z}$, $[0]_R \cap [1]_R = \emptyset$, $[0]_R \cap [2]_R = \emptyset$, and $[1]_R \cap [2]_R = \emptyset$. Prove mutually exclusive?