Intro to Number Theory

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CS 22 2020

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Overview

Divisibility (8.1)
   Facts about Divisibility (8.1.1)
   When Divisibility Goes Bad (8.1.2)
   Die Hard (8.1.3)
Definition of divides

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\( a \mid b \iff \\exists k \in \mathbb{Z} : ak = b \).

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Note: \( 2^m - 1 \) is composite if \( m \) is. (Can prove by induction!)
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- Are there infinitely many? Unknown!
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Divisibility properties

1. If $a|b$ and $b|c$, then $a|c$. 

Proof: All follow from the definition of divisibility we gave. For example: $a|b$ means $\exists k, ak = b$. Multiplying by $c$, we have $\exists k, cak = cb$. So, $ca|cb$. Also, if $ca|cb$, we have $\exists k, cak = cb$. If $c \neq 0$, that implies $\exists k, ak = b$, in other words, $a|b$. QED.

Definition: A number $n$ is a linear combination of numbers $b_0, \ldots, b_n$ iff $n = s_0 b_0 + s_1 b_1 + \cdots + s_n b_n$ for some $s_0, \ldots, s_n$. 

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Famous conjectures

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► Fermat’s Last Theorem: There are no positive integers $x$, $y$, and $z$ such that $x^n + y^n = z^n$ for some integer $n > 2$. Status: Yes, solved in 1994.
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**Theorem:** Let $n$ and $d > 0$ be integers. There exists a unique pair of integers $q$ and $r$, such that $n = q \cdot d + r$ AND $0 \leq r < d$. 
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$q = \text{qcnt}(n, d)$ is the quotient, $r = \text{rem}(n, d)$ is the remainder.
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Examples:

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- $\text{rem}(2716, 10) = 6$. Same reason.
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**Theorem:** Let $n$ and $d > 0$ be integers. There exists a unique pair of integers $q$ and $r$, such that $n = q \cdot d + r$ AND $0 \leq r < d$.

$q = \text{qcnt}(n, d)$ is the quotient, $r = \text{rem}(n, d)$ is the remainder. I’d call them “integer division” and “mod”. Languages?

Examples:

- $\text{qcnt}(2716, 10) = 271$. Since $2716 = 271 \cdot 10 + 6$
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Water jug problem

As seen in *Die Hard 3*: Given a source of water and two perfectly calibrated jugs of size 3 gallons and 5 gallons, can you measure out exactly 4 gallons?
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3. (2, 3), (5, 3), (3, 0)
4. (2, 0), (3, 3)
5. (0, 2), (5, 1)
6. (5, 2), (0, 1)
7. (4, 3), (1, 0)
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**Lemma**: With jugs of sizes $a$ and $b$, the amount of water in each jug is always a linear combination of $a$ and $b$. 
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**Base case**: In the initial state $(0,0)$, both jugs are empty, and $0$ is a linear combination of $a$ and $b$. 
**Water jug theorem**

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**Base case:** In the initial state $(0, 0)$, both jugs are empty, and 0 is a linear combination of $a$ and $b$. Specifically, $0 = 0 \cdot a + 0 \cdot b$. 
Inductive step

**Inductive step**: Suppose the state is \((x, y)\) after \(n\) moves.
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Inductive step: Suppose the state is \((x, y)\) after \(n\) moves. By our inductive hypothesis, both \(x\) and \(y\) are linear combinations of \(a\) and \(b\).
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- Empty a jug so that it contains zero gallons.
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- Fill a jug from the water source.
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- Pour water from one jug to the other until the first jug is empty. The other contains \(x + y\) gallons, which is a linear combination of \(a\) and \(b\) since both \(x\) and \(y\) were.
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- Pour water from one jug to the other until the second jug is full. The full jug contains \(a\) or \(b\). The other jug contains \(x + y - a\) or \(x + y - b\), both of which are linear combinations of \(a\) and \(b\).
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Since linear combinations are maintained, the lemma is true.