Induction Continued, Strong Induction

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Overview

Comparison of Inductions

Ordinary induction: for all $n \in \mathbb{N}$, $P(n)$ implies $P(n+1)$.

Strong induction: for all $n \in \mathbb{N}$, $P(0)$, $P(1), \ldots, P(n)$ *together* imply $P(n+1)$.

Lets you assume more in your inductive step. If you plan to use it in a proof, make sure you say so up front.

Example usage of strong induction

Theorem: Every integer greater than 1 is a product of primes.

Our proof will use strong induction on $P(n)$ "n is a product of primes".

We need to prove that $P(n)$ holds for all $n \geq 2$.

Base Case ($n = 2$): $P(2)$ is true because 2 is prime, so it is the product of (one) primes.

Inductive step

Inductive step: Suppose that $n \geq 2$ and that k is a product of primes for every integer k where $2 \leq k \leq n$. We must show that $P(n+1)$—$n+1$ is a product of primes. We proceed by cases:

- $n+1$ is prime: It is the product of (one) primes, so $P(n+1)$ holds.
- $n+1$ is not prime: By definition, $n+1 = km$ for some integers k, m such that $2 \leq k, m \leq n$. By the strong induction hypothesis, both k and m are products of primes. So, therefore, so is km and so is $n+1$. Again, $P(n+1)$ holds.

That completes the cases. And, that completes the strong induction proof.

Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

- 1? No.
- 2? No.
- 3? Yes, one 3-unit coin: $(0, 1)$.
- 4? No.
- 5? Yes, one 5-unit coin: $(1, 0)$.
- 6? Yes, two 3-unit coins: $(0, 2)$.
- 7? No.
- 8? Yes: $(1, 1)$.
- 9? Yes: $(0, 3)$.
- 10? Yes: $(2, 0)$.
- 11? Yes: $(1, 2)$.

Coin Theorem

Theorem: You can make all values 8 or larger.

Proof: We proceed by strong induction with induction hypothesis $P(n)$: There is a collection of coins whose value is n.

Base case: $P(8)$ is true because we can make 8 using $(1, 1)$.

Inductive step: Assume $P(k)$ holds for all integers $8 \leq k \leq n$, and now prove $P(n+1)$ holds. We argue by cases:

- $n = 9$: $(0, 3)$
- $n = 10$: $(2, 0)$
- $n \geq 11$: By strong induction, we can make the value $n - 2$, then add a 3-unit coin to get $n + 1$.

That completes the proof by strong induction, which proves the theorem. QED.

The game

Start with a list of numbers and a score of zero. Pick an integer z from the list and replace it with two integers x and y such that $x + y = z$. Add xy to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:

	7							0
	5					2		10
	2		3			2		16
Example:	2		2		1	2		18
	1	1	2		1	2		19
	1	1	2		1	1	1	20
	1	1	1	1	1	1	1	21

Irrelevance theorem

Theorem: No matter how you play the game starting from n, your score will be $n(n-1)/2$.

Proof: The proof is by strong induction with $P(n)$ as the proposition that every way of playing starting with the number n gives a score of $n(n-1)/2$.

Base case: Starting from a list of $n = 1$, the game ends immediately with a score of $n(n-1)/2 = 1(0)/2 = 0$. $P(1)$ is true.

Inductive step:We must show that $P(1), \ldots, P(n)$ imply $P(n+1)$ for all $n \geq 1$. Assume that $P(1), \ldots, P(n)$ are all true and we are playing starting with $n+1$. The first move must break $n+1$ into positive numbers a and b with $a + b = n + 1$. The total score for the game is the sum of points for this first move plus the scores obtained from playing with a and b:

Inductive step

$$
\begin{aligned}
\text{total score} = \quad & \text{score for turning } n+1 \text{ into } a \text{ and } b \\
& + \text{score from starting with } a \\
& + \text{score from starting with } b && \text{rules of game} \\
= \quad & ab + \tfrac{a(a-1)}{2} + \tfrac{b(b-1)}{2} && \text{inductive hyp.} \\
= \quad & \tfrac{2ab+a^2-a+b^2-b}{2} \\
= \quad & \tfrac{(a+b)^2-(a+b)}{2} && \text{algebra} \\
= \quad & \tfrac{(n+1)^2-(n+1)}{2} && \text{Defn } a,\ b \\
= \quad & (n+1)n/2 && \text{algebra}
\end{aligned}
$$

We showed $P(1), P(2), \ldots, P(n)$ implies $P(n+1)$, showing the claim true by strong induction. QED.

Well Ordering

Covered in the book, Chapter 2.

Idea: If a claim has a counterexample, it has a *least* counterexample. And, if we can prove (by contradiction) that its existence implies an even smaller counterexample, then it wasn't the smallest counterexample after all. Indeed, there must be no counterexample (!).

It is basically a rephrasing of strong induction because we're showing that being true up to some value n means it can't be false for n.

And, strong induction is just ordinary induction with a universal quantifier in its key proposition.

So, three different names for the same thing. Use is a matter of taste.