Induction Continued, Strong Induction

Michael L. Littman

CS 22 2020

February 10, 2020
Overview

Strong Induction (6.2)
   A Rule for Strong Induction (6.2.1)
   Products of Primes (6.2.2)
   Making Change (6.2.3)
   The Stacking Game (6.2.4)

Strong Induction vs. Induction vs. Well Ordering (6.3)
Comparison of Inductions

Ordinary induction: for all $n \in \mathbb{N}$, $P(n)$ implies $P(n + 1)$. 
Comparison of Inductions

Ordinary induction: for all $n \in \mathbb{N}$, $P(n)$ implies $P(n + 1)$.

Strong induction: for all $n \in \mathbb{N}$, $P(0)$, $P(1)$, \ldots, $P(n)$ together imply $P(n + 1)$.
Comparison of Inductions

Ordinary induction: for all $n \in \mathbb{N}$, $P(n)$ implies $P(n + 1)$.

Strong induction: for all $n \in \mathbb{N}$, $P(0), P(1), \ldots, P(n)$ together imply $P(n + 1)$.

Lets you assume more in your inductive step.
Comparison of Inductions

Ordinary induction: for all $n \in \mathbb{N}$, $P(n)$ implies $P(n + 1)$.

Strong induction: for all $n \in \mathbb{N}$, $P(0)$, $P(1)$, ..., $P(n)$ together imply $P(n + 1)$.

Lets you assume more in your inductive step. If you plan to use it in a proof, make sure you say so up front.
Example usage of strong induction

**Theorem:** Every integer greater than 1 is a product of primes.
Example usage of strong induction

**Theorem**: Every integer greater than 1 is a product of primes.

Our proof will use strong induction on \( P(n) \) “\( n \) is a product of primes”.

Example usage of strong induction

**Theorem:** Every integer greater than 1 is a product of primes.

Our proof will use strong induction on $P(n)$ “$n$ is a product of primes”.

We need to prove that $P(n)$ holds for all $n \geq 2$. 
Example usage of strong induction

**Theorem**: Every integer greater than 1 is a product of primes.

Our proof will use strong induction on $P(n)$ “$n$ is a product of primes”.

We need to prove that $P(n)$ holds for all $n \geq 2$.

**Base Case** ($n = 2$): $P(2)$ is true because 2 is prime, so it is the product of (one) primes.
Inductive step

**Inductive step**: Suppose that $n \geq 2$ and that $k$ is a product of primes for every integer $k$ where $2 \leq k \leq n$. 
Inductive step

**Inductive step:** Suppose that $n \geq 2$ and that $k$ is a product of primes for every integer $k$ where $2 \leq k \leq n$. We must show that $P(n+1) = n+1$ is a product of primes.
Inductive step

**Inductive step**: Suppose that $n \geq 2$ and that $k$ is a product of primes for every integer $k$ where $2 \leq k \leq n$. We must show that $P(n + 1) = n + 1$ is a product of primes. We proceed by cases:
Inductive step

**Inductive step**: Suppose that $n \geq 2$ and that $k$ is a product of primes for every integer $k$ where $2 \leq k \leq n$. We must show that $P(n+1) = n + 1$ is a product of primes. We proceed by cases:

- **$n + 1$ is prime**: It is the product of (one) primes, so $P(n+1)$ holds.
Inductive step

**Inductive step**: Suppose that $n \geq 2$ and that $k$ is a product of primes for every integer $k$ where $2 \leq k \leq n$. We must show that $P(n + 1)$—$n + 1$ is a product of primes. We proceed by cases:

- **$n + 1$ is prime**: It is the product of (one) primes, so $P(n + 1)$ holds.

- **$n + 1$ is not prime**: By definition, $n + 1 = km$ for some integers $k, m$ such that $2 \leq k, m \leq n$. 
Inductive step

**Inductive step**: Suppose that $n \geq 2$ and that $k$ is a product of primes for every integer $k$ where $2 \leq k \leq n$. We must show that $P(n + 1)$—$n + 1$ is a product of primes. We proceed by cases:

- $n + 1$ is prime: It is the product of (one) primes, so $P(n + 1)$ holds.

- $n + 1$ is not prime: By definition, $n + 1 = km$ for some integers $k$, $m$ such that $2 \leq k, m \leq n$. By the strong induction hypothesis, both $k$ and $m$ are products of primes. So, therefore, so is $km$ and so is $n + 1$. 
Inductive step

**Inductive step**: Suppose that $n \geq 2$ and that $k$ is a product of primes for every integer $k$ where $2 \leq k \leq n$. We must show that $P(n+1) = n + 1$ is a product of primes. We proceed by cases:

- **$n + 1$ is prime**: It is the product of (one) primes, so $P(n + 1)$ holds.

- **$n + 1$ is not prime**: By definition, $n + 1 = km$ for some integers $k, m$ such that $2 \leq k, m \leq n$. By the strong induction hypothesis, both $k$ and $m$ are products of primes. So, therefore, so is $km$ and so is $n + 1$. Again, $P(n + 1)$ holds.
**Inductive step**

**Inductive step**: Suppose that $n \geq 2$ and that $k$ is a product of primes for every integer $k$ where $2 \leq k \leq n$. We must show that $P(n + 1) - n + 1$ is a product of primes. We proceed by cases:

- **$n + 1$ is prime**: It is the product of (one) primes, so $P(n + 1)$ holds.

- **$n + 1$ is not prime**: By definition, $n + 1 = km$ for some integers $k$, $m$ such that $2 \leq k, m \leq n$. By the strong induction hypothesis, both $k$ and $m$ are products of primes. So, therefore, so is $km$ and so is $n + 1$. Again, $P(n + 1)$ holds.

That completes the cases.
Inductive step

**Inductive step**: Suppose that $n \geq 2$ and that $k$ is a product of primes for every integer $k$ where $2 \leq k \leq n$. We must show that $P(n + 1)$—$n + 1$ is a product of primes. We proceed by cases:

- **$n + 1$ is prime**: It is the product of (one) primes, so $P(n + 1)$ holds.

- **$n + 1$ is not prime**: By definition, $n + 1 = km$ for some integers $k, m$ such that $2 \leq k, m \leq n$. By the strong induction hypothesis, both $k$ and $m$ are products of primes. So, therefore, so is $km$ and so is $n + 1$. Again, $P(n + 1)$ holds.

That completes the cases. And, that completes the strong induction proof.
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

- 1?

- 2?

- 3? Yes, one 3-unit coin: (0, 1).

- 4?

- 5? Yes, one 5-unit coin: (1, 0).

- 6? Yes, two 3-unit coins: (0, 2).

- 7?

- 8? Yes: (1, 1).

- 9? Yes: (0, 3).

- 10? Yes: (2, 0).

- 11? Yes: (1, 2).
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

► 1? No.
► 2?
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

▶ 1? No.
▶ 2? No.
▶ 3?
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

▸ 1? No.
▸ 2? No.
▸ 3? Yes, one 3-unit coin: (0, 1).
▸ 4?

…
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

- 1? No.
- 2? No.
- 3? Yes, one 3-unit coin: (0, 1).
- 4? No.
- 5?
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

► 1? No.
► 2? No.
► 3? Yes, one 3-unit coin: (0, 1).
► 4? No.
► 5? Yes, one 5-unit coin: (1, 0).
► 6?
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

► 1? No.
► 2? No.
► 3? Yes, one 3-unit coin: (0, 1).
► 4? No.
► 5? Yes, one 5-unit coin: (1, 0).
► 6? Yes, two 3-unit coins: (0, 2).
► 7?
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

- 1? Yes, one 3-unit coin: (0, 1).
- 2? Yes, one 5-unit coin: (1, 0).
- 3? Yes, two 3-unit coins: (0, 2).
- 4? No.
- 5? Yes, three 3-unit coins: (0, 3).
- 6? Yes, two 5-unit coins: (2, 0).
- 7? Yes, one 3-unit coin and one 5-unit coin: (1, 2).
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

▶ 1? No.
▶ 2? No.
▶ 3? Yes, one 3-unit coin: (0, 1).
▶ 4? No.
▶ 5? Yes, one 5-unit coin: (1, 0).
▶ 6? Yes, two 3-unit coins: (0, 2).
▶ 7? No.
▶ 8? Yes: (1, 1).
▶ 9?
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

- 1? No.
- 2? No.
- 3? Yes, one 3-unit coin: (0, 1).
- 4? No.
- 5? Yes, one 5-unit coin: (1, 0).
- 6? Yes, two 3-unit coins: (0, 2).
- 7? No.
- 8? Yes: (1, 1).
- 9? Yes: (0, 3).
- 10?
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

- 1? No.
- 2? No.
- 3? Yes, one 3-unit coin: \((0, 1)\).
- 4? No.
- 5? Yes, one 5-unit coin: \((1, 0)\).
- 6? Yes, two 3-unit coins: \((0, 2)\).
- 7? No.
- 8? Yes: \((1, 1)\).
- 9? Yes: \((0, 3)\).
- 10? Yes: \((2, 0)\).
- 11?
Coins of 3 and 5 units

What values can you make if you have coins worth 3 and 5 units?

- 1? No.
- 2? No.
- 3? Yes, one 3-unit coin: (0, 1).
- 4? No.
- 5? Yes, one 5-unit coin: (1, 0).
- 6? Yes, two 3-unit coins: (0, 2).
- 7? No.
- 8? Yes: (1, 1).
- 9? Yes: (0, 3).
- 10? Yes: (2, 0).
- 11? Yes: (1, 2).
Coin Theorem

**Theorem:** You can make all values 8 or larger.
Coin Theorem

**Theorem**: You can make all values 8 or larger.

**Proof**: We proceed by strong induction with induction hypothesis $P(n)$: There is a collection of coins whose value is $n$. 

- **Base case**: $P(8)$ is true because we can make 8 using (1, 1).
- **Inductive step**: Assume $P(k)$ holds for all integers $8 \leq k \leq n$, and now prove $P(n+1)$ holds.
  - We argue by cases:
    - $n = 9$: (0, 3)
    - $n = 10$: (0, 3)
    - $n \geq 11$: By strong induction, we can make the value $n-2$, then add a 3-unit coin to get $n+1$.

That completes the proof by strong induction, which proves the theorem. QED.
Coin Theorem

**Theorem:** You can make all values 8 or larger.

**Proof:** We proceed by strong induction with induction hypothesis $P(n)$: There is a collection of coins whose value is $n$.

**Base case:** $P(8)$ is true because we can make 8 using (1, 1).
Coin Theorem

**Theorem:** You can make all values 8 or larger.

**Proof:** We proceed by strong induction with induction hypothesis $P(n)$: There is a collection of coins whose value is $n$.

**Base case:** $P(8)$ is true because we can make 8 using (1, 1).

**Inductive step:** Assume $P(k)$ holds for all integers $8 \leq k \leq n$, and now prove $P(n + 1)$ holds.
Coin Theorem

**Theorem:** You can make all values 8 or larger.

**Proof:** We proceed by strong induction with induction hypothesis $P(n)$: There is a collection of coins whose value is $n$.

**Base case:** $P(8)$ is true because we can make 8 using (1, 1).

**Inductive step:** Assume $P(k)$ holds for all integers $8 \leq k \leq n$, and now prove $P(n + 1)$ holds. We argue by cases:

- $n = 9$: (0, 3)
- $n = 10$: (0, 3)
- $n \geq 11$: By strong induction, we can make the value $n - 2$, then add a 3-unit coin to get $n + 1$.

That completes the proof by strong induction, which proves the theorem. QED.
Coin Theorem

**Theorem:** You can make all values 8 or larger.

**Proof:** We proceed by strong induction with induction hypothesis $P(n)$: There is a collection of coins whose value is $n$.

**Base case:** $P(8)$ is true because we can make 8 using $(1, 1)$.

**Inductive step:** Assume $P(k)$ holds for all integers $8 \leq k \leq n$, and now prove $P(n + 1)$ holds. We argue by cases:

▶ $n = 9$: $(0, 3)$
Coin Theorem

**Theorem**: You can make all values 8 or larger.

**Proof**: We proceed by strong induction with induction hypothesis $P(n)$: There is a collection of coins whose value is $n$.

**Base case**: $P(8)$ is true because we can make 8 using (1, 1).

**Inductive step**: Assume $P(k)$ holds for all integers $8 \leq k \leq n$, and now prove $P(n + 1)$ holds. We argue by cases:

- $n = 9$: (0, 3)
- $n = 10$: (0, 3)
Coin Theorem

**Theorem**: You can make all values 8 or larger.

**Proof**: We proceed by strong induction with induction hypothesis

\[ P(n) : \text{There is a collection of coins whose value is } n. \]

**Base case**: \( P(8) \) is true because we can make 8 using (1, 1).

**Inductive step**: Assume \( P(k) \) holds for all integers \( 8 \leq k \leq n \), and now prove \( P(n + 1) \) holds. We argue by cases:

\[ \begin{align*}
\& n = 9: \ (0, 3) \\
\& n = 10: \ (0, 3) \\
\& n \geq 11: \text{ By strong induction, we can make the value } n - 2, \\
\& \quad \text{then add a 3-unit coin to get } n + 1.
\end{align*} \]

That completes the proof by strong induction, which proves the theorem. QED.
The game

Start with a list of numbers and a score of zero.
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. 
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score.
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s.
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

```
7 | 0
```

Example:
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>10</td>
</tr>
</tbody>
</table>

Example:
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:

| 7  | 0  |
| 5  | 2  | 10 |
| 2  | 3  | 2  | 16 |

Example:
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:

<table>
<thead>
<tr>
<th>7</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:

\[
\begin{array}{ccc|c}
7 & & 0 \\
5 & 2 & 10 \\
2 & 3 & 16 \\
\text{Example:} & 2 & 2 & 1 & 2 & 18 \\
1 & 1 & 2 & 1 & 2 & 19 \\
\end{array}
\]
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:

<table>
<thead>
<tr>
<th>7</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Example: 2 2 1 2 18

1 1 2 1 2 19

1 1 2 1 1 1 20
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>2</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td></td>
<td>16</td>
</tr>
</tbody>
</table>

Example: 2 2 1 2 18
1 1 2 1 2 19
1 1 2 1 1 1 20
1 1 1 1 1 1 1 21
The game

Start with a list of numbers and a score of zero. Pick an integer $z$ from the list and replace it with two integers $x$ and $y$ such that $x + y = z$. Add $xy$ to your score. Continue until list consists of all 1s. Try to maximize your score.

Example:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>Example: 2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td></td>
<td>18</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td>19</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>21</td>
</tr>
</tbody>
</table>
Irrelevance theorem

**Theorem**: No matter how you play the game starting from $n$, your score will be $n(n - 1)/2$. 
Irrelevance theorem

**Theorem:** No matter how you play the game starting from $n$, your score will be $n(n - 1)/2$.

**Proof:** The proof is by strong induction with $P(n)$ as the proposition that every way of playing starting with the number $n$ gives a score of $n(n - 1)/2$. 
Irrelevance theorem

**Theorem:** No matter how you play the game starting from $n$, your score will be $n(n - 1)/2$.

**Proof:** The proof is by strong induction with $P(n)$ as the proposition that every way of playing starting with the number $n$ gives a score of $n(n - 1)/2$.

**Base case:** Starting from a list of $n = 1$, the game ends immediately with a score of $n(n - 1)/2 = 1(0)/2 = 0$. 
Irrelevance theorem

**Theorem:** No matter how you play the game starting from \( n \), your score will be \( n(n - 1)/2 \).

**Proof:** The proof is by strong induction with \( P(n) \) as the proposition that every way of playing starting with the number \( n \) gives a score of \( n(n - 1)/2 \).

**Base case:** Starting from a list of \( n = 1 \), the game ends immediately with a score of \( n(n - 1)/2 = 1(0)/2 = 0 \). \( P(1) \) is true.
Irrelevance theorem

**Theorem:** No matter how you play the game starting from $n$, your score will be $n(n - 1)/2$.

**Proof:** The proof is by strong induction with $P(n)$ as the proposition that every way of playing starting with the number $n$ gives a score of $n(n - 1)/2$.

**Base case:** Starting from a list of $n = 1$, the game ends immediately with a score of $n(n - 1)/2 = 1(0)/2 = 0$. $P(1)$ is true.

**Inductive step:**
Irrelevance theorem

**Theorem:** No matter how you play the game starting from $n$, your score will be $n(n - 1)/2$.

**Proof:** The proof is by strong induction with $P(n)$ as the proposition that every way of playing starting with the number $n$ gives a score of $n(n - 1)/2$.

**Base case:** Starting from a list of $n = 1$, the game ends immediately with a score of $n(n - 1)/2 = 1(0)/2 = 0$. $P(1)$ is true.

**Inductive step:** We must show that $P(1), \ldots, P(n)$ imply $P(n + 1)$ for all $n \geq 1$. 
Irrelevance theorem

**Theorem:** No matter how you play the game starting from \( n \), your score will be \( n(n - 1)/2 \).

**Proof:** The proof is by strong induction with \( P(n) \) as the proposition that every way of playing starting with the number \( n \) gives a score of \( n(n - 1)/2 \).

**Base case:** Starting from a list of \( n = 1 \), the game ends immediately with a score of \( n(n - 1)/2 = 1(0)/2 = 0 \). \( P(1) \) is true.

**Inductive step:** We must show that \( P(1), \ldots, P(n) \) imply \( P(n + 1) \) for all \( n \geq 1 \). Assume that \( P(1), \ldots, P(n) \) are all true and we are playing starting with \( n + 1 \).
Irrelevance theorem

**Theorem:** No matter how you play the game starting from $n$, your score will be $n(n - 1)/2$.

**Proof:** The proof is by strong induction with $P(n)$ as the proposition that every way of playing starting with the number $n$ gives a score of $n(n - 1)/2$.

**Base case:** Starting from a list of $n = 1$, the game ends immediately with a score of $n(n - 1)/2 = 1(0)/2 = 0$. $P(1)$ is true.

**Inductive step:** We must show that $P(1), \ldots, P(n)$ imply $P(n + 1)$ for all $n \geq 1$. Assume that $P(1), \ldots, P(n)$ are all true and we are playing starting with $n + 1$. The first move must break $n + 1$ into positive numbers $a$ and $b$ with $a + b = n + 1$. 


Irrelevance theorem

**Theorem:** No matter how you play the game starting from \( n \), your score will be \( n(n - 1)/2 \).

**Proof:** The proof is by strong induction with \( P(n) \) as the proposition that every way of playing starting with the number \( n \) gives a score of \( n(n - 1)/2 \).

**Base case:** Starting from a list of \( n = 1 \), the game ends immediately with a score of \( n(n - 1)/2 = 1(0)/2 = 0 \). \( P(1) \) is true.

**Inductive step:** We must show that \( P(1), \ldots, P(n) \) imply \( P(n + 1) \) for all \( n \geq 1 \). Assume that \( P(1), \ldots, P(n) \) are all true and we are playing starting with \( n + 1 \). The first move must break \( n + 1 \) into positive numbers \( a \) and \( b \) with \( a + b = n + 1 \). The total score for the game is the sum of points for this first move plus the scores obtained from playing with \( a \) and \( b \):
Inductive step

total score = score for turning $n + 1$ into $a$ and $b$
Inductive step

\[
\text{total score} = \text{score for turning } n + 1 \text{ into } a \text{ and } b \\
+ \text{score from starting with } a
\]
Inductive step

\[
\text{total score} = \text{score for turning } n + 1 \text{ into } a \text{ and } b \\
+ \text{score from starting with } a \\
+ \text{score from starting with } b \\
\]
Inductive step

\[
\text{total score} = \text{score for turning } n + 1 \text{ into } a \text{ and } b \\
\quad + \text{score from starting with } a \\
\quad + \text{score from starting with } b \\
\quad = ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2}
\]

rules of game

inductive hyp.
Inductive step

\[
\text{total score} = \text{score for turning } n + 1 \text{ into } a \text{ and } b \\
+ \text{score from starting with } a \\
+ \text{score from starting with } b \\
= ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \\
= \frac{2ab + a^2 - a + b^2 - b}{2}
\]
Inductive step

\[
\text{total score} = \text{score for turning } n + 1 \text{ into } a \text{ and } b \\
+ \text{ score from starting with } a \\
+ \text{ score from starting with } b \quad \text{rules of game} \\
= ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \quad \text{inductive hyp.} \\
= \frac{2ab+a^2-a+b^2-b}{2} \\
= \frac{(a+b)^2-(a+b)}{2} \quad \text{algebra}
\]
Inductive step

\[
\text{total score} = \text{score for turning } n + 1 \text{ into } a \text{ and } b \\
\quad + \text{ score from starting with } a \\
\quad + \text{ score from starting with } b \\
\quad = ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \\
\quad = \frac{2ab + a^2 - a + b^2 - b}{2} \\
\quad = \frac{(a+b)^2 - (a+b)}{2} \\
\quad = \frac{(n+1)^2 - (n+1)}{2}
\]

rules of game
inductive hyp.

algebra
Defn \(a, b\)
Inductive step

total score = score for turning $n + 1$ into $a$ and $b$
+ score from starting with $a$
+ score from starting with $b$
    \[ \begin{align*}
    &= ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \\
    &= \frac{2ab + a^2 - a + b^2 - b}{2} \\
    &= \frac{(a+b)^2 - (a+b)}{2} \\
    &= \frac{(n+1)^2 - (n+1)}{2} \\
    &= (n + 1) n / 2
    \end{align*} \]
rules of game
inductive hyp.
algebra
Defn $a$, $b$
algebra
Inductive step

\[
\text{total score} = \text{score for turning } n+1 \text{ into } a \text{ and } b \\
+ \text{ score from starting with } a \\
+ \text{ score from starting with } b \\
= ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \\
= \frac{2ab + a^2 - a + b^2 - b}{2} \\
= \frac{(a+b)^2 - (a+b)}{2} \\
= \frac{(n+1)^2 - (n+1)}{2} \\
= (n + 1)n/2
\]

We showed \( P(1), P(2), \ldots, P(n) \) implies \( P(n + 1) \), showing the claim true by strong induction. QED.
Well Ordering

Covered in the book, Chapter 2.
Well Ordering

Covered in the book, Chapter 2.

**Idea:** If a claim has a counterexample, it has a *least* counterexample. And, if we can prove (by contradiction) that its existence implies an even smaller counterexample, then it wasn’t the smallest counterexample after all. Indeed, there must be no counterexample (!).
Well Ordering

Covered in the book, Chapter 2.

**Idea**: If a claim has a counterexample, it has a *least* counterexample. And, if we can prove (by contradiction) that its existence implies an even smaller counterexample, then it wasn’t the smallest counterexample after all. Indeed, there must be no counterexample (!).

It is basically a rephrasing of strong induction because we’re showing that being true up to some value $n$ means it can’t be false for $n$. 

Well Ordering

Covered in the book, Chapter 2.

**Idea**: If a claim has a counterexample, it has a least counterexample. And, if we can prove (by contradiction) that its existence implies an even smaller counterexample, then it wasn’t the smallest counterexample after all. Indeed, there must be no counterexample (!).

It is basically a rephrasing of strong induction because we’re showing that being true up to some value $n$ means it can’t be false for $n$.

And, strong induction is just ordinary induction with a universal quantifier in its key proposition.
Well Ordering

Covered in the book, Chapter 2.

**Idea**: If a claim has a counterexample, it has a *least* counterexample. And, if we can prove (by contradiction) that its existence implies an even smaller counterexample, then it wasn’t the smallest counterexample after all. Indeed, there must be no counterexample (!).

It is basically a rephrasing of strong induction because we’re showing that being true up to some value $n$ means it can’t be false for $n$.

And, strong induction is just ordinary induction with a universal quantifier in its key proposition.

So, three different names for the same thing. Use is a matter of taste.