Set Equality & Quantification

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Overview

Proof by Cases (1.7)

Predicate Formulas (3.6)
Fact about groups of people

Any two people have either met or not.
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Given a group of people $G$, if all pairs of people in $G$ have met, we’ll call it a \textit{club}. If no two people in $G$ have met, we’ll call them \textit{strangers}.
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**Theorem.** Every collection of 6 people includes a club of 3 people or a group of 3 strangers.
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**Theorem.** Every collection of 6 people includes a club of 3 people or a group of 3 strangers.

Does that seem true? Try some examples on the board.
Proof (Part 1)

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1. Among $R$, at least 3 have met $x$. 

2. Among $R$, at least 3 have not met $x$. 

At least one of these cases must hold. Since $|R|$ is odd, either more than half in $R$ know $x$ or less than half in $R$ know $x$ (and therefore more than half do not know $x$).

Case 1: At least 3 have met $x$. Let $J \subseteq R$ be those individuals.

Two subcases:

1.1 No pair in $J$ have met each other. So, $J$ is a group of at least 3 strangers and the theorem holds in this subcase.

1.2 Some pair in $J$ have met each other. That pair and $x$ are a club of 3 people and the theorem holds in this subcase, too.

That covers Case 1!
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That covers Case 1!
Case 2: At least 3 have not met x. Let $J \subseteq R$ be those individuals. Two subcases:
Proof (Part 2)

Case 2: At least 3 have not met $x$. Let $J \subseteq R$ be those individuals. Two subcases:

2.1 Every pair in $J$ have met each other. So, $R$ is a club of at least size 3 and the theorem holds in this subcase.
Proof (Part 2)

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Since we showed that only these two cases can occur and the theorem holds in both, the theorem always holds.
Quantifiers, Revisited

**Always True** (universal quantification)

\[ \forall x \in \mathbb{R}, x^2 + 1 \geq 0. \]
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- There is an \( x \in D \) such that \( P(x) \) is true.
- \( P(x) \) is true for some \( x \) in the set \( D \).
- \( P(x) \) is true for at least one \( x \in D \).
Mixing quantifiers

**Theorem** (sparse squares): There’s a perfect square arbitrarily far from its closest perfect square.
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$$\forall d \in \mathbb{N}, \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, \text{ } i \text{ is a perfect square AND } |i - j| \leq d \text{ IMPLIES } j \text{ is NOT a perfect square.}$$
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You can think of it like a little game. I’m claiming that you can pick any \( d \) you want. I’ll then pick an \( i \) that’s a perfect square AND no matter what \( j \) you pick that is within \( d \) values of \( i \), \( j \) won’t be a perfect square.
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So, what’s my winning strategy?
Any ambiguity is too many

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1. If $\exists o$, you can juggle $o$, then you’ve got a talent.
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“...statistics show that, in New York, a man is mugged every 11 seconds. I would now like you to meet that man. His name is Jesse Donnally.”
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1. \( \forall t, \exists m, m \) mugged at time \( t \)
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From my files

Addressing a group: “Send me all of your papers.”
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\[ \forall x \text{ in group}, \forall \text{ papers } p, x \text{ wrote } p \implies \text{send}(x) \]
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Addressing a group: “Send me all of your papers.”
- ∀x in group, ∀papers p, x wrote p IMPLIES send(x)
- ∀papers p, (∀x in group, x wrote p) IMPLIES send(x)

About a medical side effect: “Everything tastes the same”
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- \( \forall x, \forall y, \text{taste}(x, \text{ now}) = \text{taste}(y, \text{ now}) \)
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“The whole article is not available.”

- $\neg \forall \text{ article part } x, x \text{ is available}$
- $\forall \text{ article part } x, x \text{ is not available}$
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DeMorgan returns: Negating quantifiers

These two statements are equivalent:
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These two statements are equivalent:

► Not everyone likes chocolate.
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These two statements are equivalent:

▶ Not everyone likes chocolate.
▶ There’s someone who doesn’t like chocolate.
DeMorgan returns: Negating quantifiers

These two statements are equivalent:

- Not everyone likes chocolate.
- There’s someone who doesn’t like chocolate.

\[ \neg \forall x, P(x) \text{ is equivalent to } \exists x, \neg P(x). \]
Assertion about predicates

\[ \exists x, \forall y, P(x, y) \text{ IMPLIES } \forall y, \exists x, P(x, y). \]
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If \( \exists x, \forall y, P(x, y) \), there must be some specific \( x^* \) such that \( \forall y, P(x^*, y) \).
Assertion about predicates

$$\exists x, \forall y, P(x, y) \text{ IMPLIES } \forall y, \exists x, P(x, y).$$

If $$\exists x, \forall y, P(x, y),$$ there must be some specific $$x^*$$ such that $$\forall y, P(x^*, y).$$ As a result, $$\forall y, \exists x, P(x, y)$$ because we can always choose $$x^*$$ to be the selected $$x.$$
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On the other hand,
\( \forall y, \exists x, P(x, y) \text{ IMPLIES } \exists x, \forall y, P(x, y) \)
is not true.
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\[
\begin{array}{ccc}
  \text{y}_1 & \text{y}_2 & \text{y}_3 \\
  \text{x}_1 & T & F & F \\
  \text{x}_2 & T & T & T \\
  \text{x}_3 & F & T & F
\end{array}
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