Set Equality & Quantification

Michael L. Littman

CS 22 2020

January 29, 2020

Overview

Proof by Cases (1.7)

Predicate Formulas (3.6)

Fact about groups of people

Any two people have either met or not.

Given a group of people G, if all pairs of people in G have met, we'll call it a *club*. If no two people in G have met, we'll call them *strangers*.

Theorem. Every collection of 6 people includes a club of 3 people or a group of 3 strangers.

Does that seem true? Try some examples on the board.

Proof (Part 1)

The proof is by case analysis. Let x denote one of the six people. Let $R = G - \{x\}$ be the rest. There are two cases:

1. Among R, at least 3 have met x.
2. Among R, at least 3 have *not* met x.

At least one of these cases must hold. Since $|R|$ is odd, either more than half in R know x or less than half in R know x (and therefore more than half do not know x).

Case 1: At least 3 have met x. Let $J \subseteq R$ be those individuals. Two subcases:

1.1 No pair in J have met each other. So, J is a group of at least 3 strangers and the theorem holds in this subcase.

1.2 Some pair in J have met each other. That pair and x are a club of 3 people and the theorem holds in this subcase, too.

That covers Case 1!

Proof (Part 2)

Case 2: At least 3 have not met x. Let $J \subseteq R$ be those individuals. Two subcases:

2.1 Every pair in J have met each other. So, R is a club of at least size 3 and the theorem holds in this subcase.

2.2 Some pair in J haven't met each other. That pair and x are a group of strangers of 3 people and the theorem holds in this subcase, too.

That covers Case 2! It's kind of the inverse-video version of Case 1.

Since we showed that only these two cases can occur and the theorem holds in both, the theorem *always* holds.

Quantifiers, Revisited

Always True (universal quantification)
$\forall x \in \mathbb{R}, x^2 + 1 \geq 0$.

- For all $x \in D$, $P(x)$ is true.
- $P(x)$ is true for every x in the set D.

Sometimes True (existential quantification)
$\exists x \in \mathbb{Z}, x$ is even and x is prime.

- There is an $x \in D$ such that $P(x)$ is true.
- $P(x)$ is true for some x in the set D.
- $P(x)$ is true for at least one $x \in D$.

Mixing quantifiers

Theorem (sparse squares): There's a perfect square arbitrarily far from its closest perfect square.

Clear? Maybe a tad vague. True? How say in math?

$\forall d \in \mathbb{N}, \exists i \in \mathbb{N}, \forall j \in \mathbb{N}$, i is a perfect square AND $|i - j| \leq d$ IMPLIES j is NOT a perfect square.

The expressions nest inside each other. The order matters.

You can think of it like a little game. I'm claiming that you can pick any d you want. I'll then pick an i that's a perfect square AND no matter what j you pick that is within d values of i, j won't be a perfect square.

So, what's my winning strategy?

Any ambiguity is too many

"If you can juggle any object, you've got a talent."

1. If $\exists o$, you can juggle o, then you've got a talent.
2. If $\forall o$, you can juggle o, then you've got a talent.

"...statistics show that, in New York, a man is mugged every 11 seconds. I would now like you to meet that man. His name is Jesse Donnally."

1. $\forall t, \exists m$, m mugged at time t
2. $\exists m, \forall t$, m mugged at time t

From my files

Addressing a group: "Send me all of your papers."

- $\forall x$ in group, $\forall$ papers p, x wrote p IMPLIES send(x)
- $\forall$ papers p, ($\forall x$ in group, x wrote p) IMPLIES send(x)

About a medical side effect: "Everything tastes the same"

- $\forall x, \forall y$, taste(x, now) $=$ taste(y, now)
- $\forall x$, taste(x, now) $=$ taste(x, then)

"The whole article is not available."

- not $\forall$ article part x, x is available
- $\forall$ article part x, x is not available

DeMorgan returns: Negating quantifiers

These two statements are equivalent:

- Not everyone likes chocolate.
- There's someone who doesn't like chocolate.

not $\forall x, P(x)$ is equivalent to $\exists x$, not $P(x)$.

Assertion about predicates

$\exists x, \forall y, P(x, y)$ IMPLIES $\forall y, \exists x, P(x, y)$.

If $\exists x, \forall y, P(x, y)$, there must be some specific x^* such that $\forall y, P(x^*, y)$. As a result, $\forall y, \exists x, P(x, y)$ because we can always choose x^* to be the selected x.

On the other hand,
$\forall y, \exists x, P(x, y)$ IMPLIES $\exists x, \forall y, P(x, y)$
is not true.

	y_1	y_2	y_3
x_1	T	F	F
x_2	T	T	T
x_3	F	T	F

	y_1	y_2	y_3
x_1	T	F	F
x_2	T	F	T
x_3	F	T	F