Intro to Set Theory

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Overview

Sets Definitions (4.1–4.1.1)

Sets Operations (4.1.2–4.1.5)

Equality Proofs (4.1.6)

Venn Diagrams

Set Definition

Definition (informal): A *set* is a bunch/collection/group of objects.

Definition: The *elements* of the set are the objects contained in that set.

Sets can contain numbers, ordered sequences of numbers, strings, names, or other sets.

Objects are either *in* the set or *not in* the set. We don't have a concept of an object being in a set multiple times. It's a Boolean property.

We write curly braces around a comma-separated list to build a set.

Examples:

- $H = \{$ Julie, Tyler, Julia $\}$
- $D = \{$ Boston Kreme, Glazed, Apple Crumb, Pumpkin $\}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
- $S = \{$ Brown, Columbia, Cornell, ..., Yale $\}$

Elements

- $H = \{$ Julie, Tyler, Julia $\}$
- $D = \{$ Boston Kreme, Glazed, Apple Crumb, Pumpkin $\}$
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
- $S = \{$ Brown, Columbia, Cornell, …, Yale $\}$

Definition: We say say $x \in S$ if x is *an element of* or *in* or *a member of* the set S.

- Julie $\in H$? Yes.
- Columbia $\in H$? No. Columbia $\notin H$.
- Columbia $\in S$? Yes.
- Dartmouth $\in S$? Yes.

Some Sets of Numbers

- $\emptyset = \{\}$ (empty set, null set)
- $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ (non-negative integers)
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integers)
- $\mathbb{Q} = \{1/2, -4/15, 21, \ldots\}$ (rationals)
- $\mathbb{R} = \{\sqrt{2}, -\pi, 21, \ldots\}$ (real numbers)
- $\mathbb{C} = \{i/2, 15 - i, \sqrt{7}, 21, \ldots\}$ (complex numbers)

Superscript plus limits to positive values: $\mathbb{Z}^+ = \mathbb{N}^+$.

Superscript minus limits to negative values: $21 \notin \mathbb{R}^-$.

Sets of sets

- $A = \{1, 4, 9\}$
- $B = \{\{1, \{4\}\}, \{9\}\}$

- $1 \in A$? Yes.
- $1 \in B$? No, but $\{1, \{4\}\} \in B$.
- $\exists x \in B, 1 \in x$? Yes, $x = \{1, \{4\}\} \in B$ and $1 \in x$.

Subsets

Definition: One set is a *subset* of another if every element of the first set is also an element of the second.

We write $S \subseteq T$ to say the set S is a subset of set T. So, $S \subseteq T$ means $\forall x \in S, x \in T$.

Examples:

- $\mathbb{N} \subseteq \mathbb{Z}$? Yes, every positive integer is also a non-negative integer.
- $\mathbb{Z}^+ \subseteq \mathbb{N}$? Yes, every positive integer is also a non-negative integer.
- $\mathbb{C} \subseteq \mathbb{Z}$? No, $\mathbb{C} \not\subseteq \mathbb{Z}$. Some (many!) complex numbers are not integers. Although, $\mathbb{Z} \subseteq \mathbb{C}$.
- $\mathbb{N} \subseteq \mathbb{N}$. Yes, if sets are equal, all of the first must also be in the second!

Note: $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ looks a little bit like $3 \leq 4$.
We write $A \subset B$ to rule out equality (like $a < b$).

Non-Trichotomy

If a and b are integers, exactly one of these properties must hold:

- $a < b$
- $a = b$
- $a > b$

Not so for subsets. Example?
$A = \{0\}$
$B = \{1\}$
$A \subset B$? No, $0 \in A$, but $0 \notin B$.
$A = B$? No, they are different sets.
$A \supset B$? (superset!) No, $1 \in B$, but $1 \notin A$.

Operations on sets: Union

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The *union* of sets X and Y, $X \cup Y$, consists of every element that is in either X or Y. In other words, $z \in X \cup Y$ means $z \in X$ or $z \in Y$.

Example: $A \cup B = \{j, u, l, i, e, a\}$.

Operations on sets: Intersection

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The *intersection* of sets X and Y, $X \cap Y$, consists of every element that is in both X and Y. In other words, $z \in X \cap Y$ means $z \in X$ and $z \in Y$.

Example: $B \cap C = \{l, e\}$.

Operations on sets: Set difference

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The *set difference* of sets X and Y, $X - Y$, consists of every element that is in X but not in Y. In other words, $z \in X - Y$ means $z \in X$ and $z \notin Y$.

Example: $C - A = \{t, y, e, r\}$.

Example: $A - B = \{a\}$.

Operations on sets: Symmetric difference

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The *symmetric difference* of sets X and Y, $X \triangle Y$, consists of every element that is in X but not in Y or in Y but not X. In other words, $z \in X \triangle Y$ means $z \in X$ and $z \notin Y$ or $z \in Y$ and $z \notin X$.

Example: $C \triangle A = \{t, y, e, r, j, u, i, a\}$.

Example: $A \triangle B = \{a, e\}$.

Operations on sets: Complement

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The *complement* of a set X, $\overline{X}$, is defined with respect to some universe of possible elements U. It consists of every possible element that is not X. In other words, $\overline{X} ::= U - X$.

Example: If U is the universe of all letters in English, $\overline{A} = \{b, c, d, e, f, g, h, k, m, n, o, p, q, r, s, t, v, w, x, y, z\}$.

Example: If $U = \mathbb{Z}$, $\mathbb{Z}^- = \overline{\mathbb{Z}^+} - \{0\}$.

Disjoint sets

Definition: Sets A and B are *disjoint* is they have no elements in common. In other words,

$A \cap B = \emptyset$ or

$A \subseteq \overline{B}$.

Operations on sets: Power set

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The *power set* of a set X, $\mathcal{P}(X)$, is the set of all subsets of X. In other words, $\forall x \in \mathcal{P}(X), x \subseteq X$ and $\forall x \subseteq X, x \in \mathcal{P}(X)$.

Example: If $D = B \cap C = \{l, e\}$, $\mathcal{P}(D) = \{\{\}, \{l\}, \{e\}, \{l, e\}\}$.

Example: If $E = A \cap B - C = \{j, u, i\}$, $\mathcal{P}(E) = \{\{\}, \{j\}, \{u\}, \{i\}, \{j, u\}, \{j, i\}, \{u, i\}, \{j, u, i\}\}$.

Example: $\mathcal{P}(\emptyset) = \{\emptyset\}$.

Operations on sets: Cardinality

- $A = \{j, u, l, i, a\}$
- $B = \{j, u, l, i, e\}$
- $C = \{t, y, l, e, r\}$

Definition: The *cardinality* of a set X, $|X|$, is the count of the number of unique elements in X.

Example: $|A| = |B| = |C| = 5$.

Example: $|\emptyset| = 0$.

Example: If $|A| = n$, $|\mathcal{P}(A)| = 2^n$. Each subset consists of a decision of whether to include or not include (2 possibilities) each of the n elements of A.

Building sets with predicates

General form: { description of a set | filter on the set }.
Examples:

- $A ::= \{n \in \mathbb{N} | n = 2k + 1 \text{ for some integer } k\}$
- $B ::= \{x \in \mathbb{R} | x^2 > 1\}$

Note: Python has a notation for this idea.

Proving set equalities: Logic

$A = B$ means that, for all x, $x \in A$ if and only if $x \in B$. We can use this definition to prove various set equalities. Here's a useful one.

Theorem: $A = B$ if and only if (iff) both $A \subseteq B$ and $B \subseteq A$.

$A = B$
iff for all x, $x \in A$ if and only if $x \in B$
iff for all x, if $x \in A$ then $x \in B$ and for all x, if $x \in B$ then $x \in A$
iff $\forall x \in A, x \in B$ and $\forall x \in B, x \in A$
iff $A \subseteq B$ and $B \subseteq A$.

Proving set equalities: set-element method

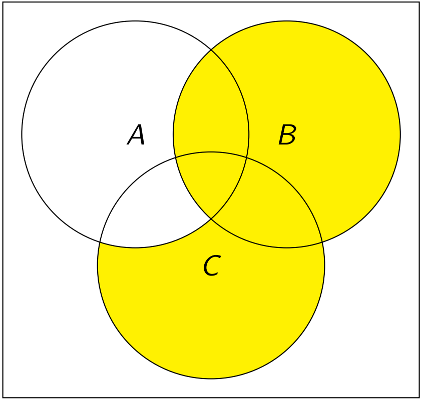
We can use the previous result to prove other set equalities.

Theorem: For any sets A and B of elements in universe U,
$\overline{A \cap B} = \overline{A} \cup \overline{B}$.

"DeMorgan's Law" relates intersection and union ("and" and "or").

An object $x \in \overline{A \cap B}$ if it is *not* in both A and B. Such an element must either not be in A or not be in B. It follows that such an element must be in $\overline{A} \cup \overline{B}$. Thus, we have $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.
Note also that an object $x \in \overline{A} \cup \overline{B}$ if it is either not in A or it is not in B. Such an element can't be in both A and B, therefore. Said another way, it must be in $\overline{A \cap B}$. Thus, we have $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.
Since both $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ are true, we know
$\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Venn diagram translation



$(C - A) \cup B$

$(C - A - B) \cup (C \cap B - A) \cup (A \cap B \cap C) \cup (A \cap B - C) \cup (B - A - C)$