Strong Induction

Requirements

1. Formally **define the predicate** that will be proved inductively.
2. Prove that the predicate holds in the **base case**.
3. Formally state the **inductive hypothesis**.
4. Assume the inductive hypothesis, and prove the **inductive step**.
5. **Conclude** that the predicate holds in general.

Example

Prove that every integer $n \geq 2$ can be written as a product of one or more prime numbers.

Proof

Let $P(n)$ be the predicate "n can be written as a product of one or more prime numbers".

Base case. The integer 2 is prime, so it is a product of exactly one prime number (itself). Therefore, $P(2)$ is true.

Inductive Hypothesis. Assume the inductive hypothesis, that for a particular k, $P(i)$ is true for all $2 \leq i \leq k$.

Inductive Step. We must prove $P(k+1)$, that $k+1$ is the product of one or more prime numbers. $k+1$ is either prime or composite. If it is prime, then it is the product of exactly one prime number (itself), and $P(k+1)$ is true. If it is composite, then by definition it is the product of two factors, $k+1 = ab$, where a and b are integers ≥ 2. Since a and b are both greater than 1, they must also both be less than $k+1$. By the inductive hypothesis, a and b can each by written as a product of one or more primes. But since $k+1 = ab$, we can combine these two products to express $k+1$ as a product of primes, so $P(k+1)$ is true.

Conclusion. Since $P(2)$ is true and $P(2), \ldots, P(k)$ together imply $P(k+1)$, $P(n)$ is true for all integers $n \geq 2$. $\square$