More Induction Proofs

Example 1

Prove that $f(n) = 6n^2 + 2n + 15$ is odd for all $n \in \mathbb{Z}^+$.

Proof by induction:
Define $P(n)$ as the predicate that $f(n)$ is odd.

Base Case. We prove $P(1)$.
$2(1) + 15 + 6(1)^2 = 2 + 15 + 6 = 23$. Since 23 is odd, $P(1)$ is true.

Inductive Hypothesis. Assume $P(k)$ for some $k \in \mathbb{Z}^+$.
That means that $16k^2 + 2k + 15 = 2m + 1$ for some $m \in \mathbb{Z}$.

Inductive Step. We will now prove $P(k+1)$.

$$\begin{aligned} 6(n+1)^2 + 2(n+1) + 15 &= (6n^2 + 12n + 6) + (2n + 2) + 15 \\ &= (2n + 15 + 6n^2) + 12n + 8 \\ &= (2m + 1) + 12n + 8 \\ &= 2(m + 6n + 4) + 1 = 2j + 1 \end{aligned}$$

Since $j = m + 6n + 4 \in \mathbb{Z}$, $f(k+1)$ is odd. Thus, $P(k) \Rightarrow P(k+1)$.
Since $P(1)$ and $P(k) \Rightarrow P(k+1), P(k)$ for all $k \in \mathbb{Z}^+$.

Example 2

Prove that if $x \geq -1$, then $(1+x)^n \geq 1 + nx$ for all integers $n \geq 1$.

Proof by induction:
Let $P(n)$ be that statement that $(1+x)^n \geq 1 + nx$ for all $x \geq -1$.

Base Case. First, we prove $P(1)$.
$(1+x)^1 \geq 1 + x$ for all $x \geq -1$. Therefore $P(1)$ holds.

Inductive Hypothesis. Assume that $P(k)$ is true for some natural number $k \geq 1$.

Inductive Step. We will prove $P(k+1)$.
From the IH, we know that for all $x \geq -1$, $(1+x)^k \geq 1 + kx$.
Multiplying both sides by $1 + x \geq 0$ gives:

$$\begin{aligned} (1+x)^{k+1} &\geq (1+kx)(1+x) \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x \end{aligned}$$

So $P(k) \Rightarrow P(k+1)$, and the inductive step is proved.
Because $P(1)$ is true and $P(k)$ implies $P(k+1)$ for all $k \geq 1$, $P(n)$ is true for all $n \geq 1$.

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Example 3

The substrings of a string $a_1a_2\ldots a_n$ are all strings of the form $a_ia_{i+1}\ldots a_j$ for all i, j such that $0 \leq i \leq j \leq n$ and where a_0 is the empty string. For example, "", "ba", and "banana" are substrings of the string "banana," while "bann" is not. "b" can be represented as either a_0a_1 or a_1, but these different representations are not distinct substrings. "an" can be represented as either a_1a_2 or a_3a_4; these are distinct substrings.

Prove that the number of distinct substrings of a string $a_1a_2\ldots a_n$ is

$$\frac{n(n+1)}{2} + 1.$$

Proof by induction:

Let $P(k)$ be the predicate "all strings $a_1a_2\ldots a_k$ have exactly $\frac{n(n+1)}{2} + 1$ distinct substrings."

Base Case. We show $P(0)$.
A string of length 0 is simply the empty string a_0. The only substring of the empty string is the empty string, meaning that there is only one substring. $\frac{0*1}{2} + 1 = 1$. Thus, $P(0)$ is true.

Inductive Hypothesis. Assume that $P(k)$ is true for some $k \geq 0$; that is, for all strings $a_1a_2\ldots a_k$ there are exactly $\frac{k(k+1)}{2} + 1$ distinct substrings.

Inductive Step. Now $P(k+1)$ must be shown to hold, meaning that for all strings $a_1a_2\ldots a_ka_{k+1}$ there are exactly $\frac{(k+1)(k+2)}{2} + 1$ substrings.

The string $a_1a_2\ldots a_ka_{k+1}$ has exactly one more character than some string $a_1a_2\ldots a_k$. All substrings of the shorter string are substrings of the longer, and for the longer there are also $k+1$ distinct new substrings: all substrings of the shorter string that end in a_k with a_{k+1} added to the end, as well as just a_{k+1}.

By inductive hypothesis, there are $\frac{k(k+1)}{2} + 1$ substrings of the shorter string; thus, there are $\frac{k(k+1)}{2} + 1 + k + 1$ substrings of the longer.

$$\begin{aligned}
\frac{k(k+1)}{2} + 1 + k + 1 &= \frac{k(k+1)}{2} + 1 + \frac{2(k+1)}{2} \\
&= \frac{k(k+1) + 2(k+1)}{2} + 1 \\
&= \frac{k^2 + 3k + 2}{2} + 1 \\
&= \frac{(k+1)(k+2)}{2} + 1
\end{aligned}$$

This is what was expected and required; clearly the number of substrings of all strings $a_1a_2\ldots a_ka_{k+1}$ is $\frac{(k+1)(k+2)}{2} + 1$, so $P(k)$ implies $P(k+1)$.

Thus, because $P(0)$ is true and $P(k)$ implies $P(k+1)$ for all $k \geq 0$, $P(n)$ is true for all $n \geq 0$.

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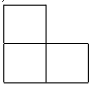
Example 4

Prove that for $n \in \mathbb{Z}^+$, a $2^n \times 2^n$ chessboard with any one square removed can be tiled by these 3-square L-tiles. 

Proof by induction:
Let $P(n)$ be the predicate that a $2^n \times 2^n$ chessboard with any one square removed can be tiled by the 3-square L-tiles.

Base Case. We prove $P(1)$.
For $n = 1$, we have a 2×2 board with 1 square removed, which can be tiled by 1 L-tile.

Inductive Hypothesis. Assume $P(k)$ holds for some $k \geq 1$.

Inductive Step. We prove $P(k + 1)$.
Divide the $2^{k+1} \times 2^{k+1}$ board as follows, where A,B,C,D are each a $2^k \times 2^k$ board.

A	B
C	D

Without loss of generality, suppose that the one square has been removed from B.
Then by the inductive hypothesis, B can be tiled.

Remove the center corners of A,C, and D, such that each of their remainders can be tiled by the inductive hypothesis. Tile the remaining 3 squares in the center with a single L-tile, and we have completed the tiling.

Since the base case $P(1)$ holds and we have shown that $P(k) \Rightarrow P(k + 1)$, $P(n)$ holds for $n \geq 1$. QED.