Midterm 2 Review Problems

Due: Never

All homeworks are due at 12:55 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

Consider a string of size $n \geq 2$, $S = s_1s_2...s_n$, where each $s_i$ is a 1-9 digit. For each of the following conditions, find the number $N$ of such strings that satisfy the condition, and prove your answer.

a. Only $x$ types of digits are used, where $1 \leq x \leq 9$.

b. No two consecutive digits are the same.

c. The sum of any $k$ consecutive digits is the same, where $1 \leq k \leq n$.

d. The product of any set of $k$ consecutive digits is the same, where $1 \leq k \leq n$.

e. The sum of all the digits is 9.

f. The digit 1 appears exactly 4 times.

g. Let $n = 9$. There are no repeated digits, and at least one of the sequences 19, 41, or 74 appears.

Problem 2

a. Add parentheses to the following expressions to make them true (1 represents true, and 0 represents false). Note also that there is no implicit ordering; that is, all ordering comes from your parentheses. State explicitly any assumptions you are making about the order of operations.

i. $0 \land 1 \lor 1 \Rightarrow 1 \land 1 \land 1 \lor 0$

ii. $0 \lor 0 \land 1 \land 0 \land 1 \land 1 \lor 1$

iii. $0 \land 1 \lor 1 \Leftrightarrow 0 \Rightarrow 0$

iv. $0 \lor 1 \Rightarrow 0 \land 0 \Rightarrow 1$

b. Prove the following logical equivalences:
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i. \( \neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q \)
ii. \( (p \Rightarrow r) \land (q \Rightarrow r) \equiv (p \lor q) \Rightarrow r \)

c. Prove that \( q \land \neg(p \Rightarrow q) \) is unsatisfiable.
d. Prove that \( (p \land q) \Rightarrow (p \lor q) \) is valid.

Problem 3

The CS022 TA’s make a good problem with an 80% probability and a bad problem with a 20% probability. The students either like or dislike each problem. If it was a good problem, 60% of the students like it. If it was a bad problem, 95% of the students dislike it.

a. What is the probability that a student will like a problem?
b. Given that a student dislikes a problem, what’s the probability that it was actually a bad problem?

Problem 4

a. Although we have been using AND and OR gates with only two inputs, we can create AND and OR gates that take in more than two inputs. Under the hood, these can just be implemented with 2-input AND and OR gates. Build a 3-input AND circuit using only 2-input AND gates, and build a 3-input OR circuit using only 2-input OR gates.
b. Duncan and Tim have spent many nights arguing about who makes better donuts. They decide to create a poll to determine what people think. However, to do so, they need a machine that can handle the poll data.

To help them out, construct a circuit using only AND, OR, and NOT gates. You may use gates of any number of inputs for this problem.

The circuit will take in three people’s answers (1 for Duncan and 0 for Tim), and return the answer of the majority.
c. Kristy and Kareem are currently having a lot of trouble determining if the numbers 0, 1, 2, and 3 are odd or even. The solution? Build a circuit!

Model this problem as a circuit using only AND, OR, and NOT gates. You may use gates with more than one input. Create three input wires and one output wire. If an even number of inputs are on, the output should be off. If an odd number of input wires are on, the output should be on.
Note: The output depends only on how many of the input wires are on. For example, if exactly one input is on, then the output should be on. It does not matter which input is on.

Problem 5

a. On a Cartesian grid with height and width equal to \( n \), if you can only travel on the horizontal and vertical gridlines, how many paths of shortest length go from the point \((0,0)\) to \((n,n)\)?

b. Now, holes have appeared in the grid at any point \((x,y)\) such that \( x < y \). With the same rules as before, how many paths of shortest length go from \((0,0)\) to \((n,n)\) without hitting the holes in the grid?

c. Another Cartesian grid has height \( n - 1 \) and width \( n + 1 \), but has holes in the grid at all points \((x,y)\) with \( x \leq y \) (except for \((0,0)\)). Prove that this new grid has the same number of valid paths as the one in part b by showing the existence of a bijection between them.

Hint: Drawing a diagram to familiarize yourself with valid paths can be helpful. Diagrams are an acceptable supplement to a proof, but never a replacement for one.

Problem 6

a. Let \( S = \{1,2,\ldots,2n\} \) for some integer \( n \). Show that for any \( T \subset S \) such that \( |T| = n + 1 \), there are elements \( x, y \in T \) such that \( x \) and \( y \) are relatively prime.

b. A repunit is a number that contains only the number 1 (1, 11, 111, 1111, etc.). Prove, using the pigeonhole principle, that among the first 50 repunits, at least one of them is divisible by 49.

Problem 7

After a thrilling day in CS22, Donny finds himself craving a donut from PVDonuts. Walking down Ives, he miraculously finds a stack of coupons for PVDonuts, and thinks it must be his lucky day. Little does he know that the coupons don’t all decrease the price of the donut: each coupon changes the price by a factor of \( \frac{1}{2}, 2 \) or 4 with equal probability.

a. What is the expected value of Donny’s donut price after using one coupon? Assume that the price is initially 1 dollar.
b. Donny uses some number of coupons in a sequence. How many ways can Donny return to his original donut price after using 4 coupons?

c. How many ways can Donny return to his original donut price after using 6 coupons?

d. What is the probability that Donny will return to his original donut price after using 6 coupons?

Problem 8

For each of the following, answer the questions and provide your reasoning.

a. Can \text{AND} (\land) be expressed using only \text{OR} (\lor) and \text{NOT} (\neg)?

b. A set of Boolean operations is \text{frosted} if all possible truth tables can be expressed using only the operators in that set. Is the set \{\text{OR, AND}\} frosted?

c. Is the set \{\text{OR, AND, NOT}\} frosted?

d. Consider the \spadesuit operator, defined by \( p \spadesuit q = \neg (p \lor q) \). Is the set \{\spadesuit\} frosted?

Problem 9

SET is an \textit{atrociously fun} card game with 81 distinct cards. Every card has four characteristics: Color, Shape, Number, and Pattern. Possible options for each characteristic are listed below:

<table>
<thead>
<tr>
<th>Color</th>
<th>Shape</th>
<th>Number</th>
<th>Pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>red</td>
<td>oval</td>
<td>1</td>
<td>solid</td>
</tr>
<tr>
<td>green</td>
<td>diamond</td>
<td>2</td>
<td>striped</td>
</tr>
<tr>
<td>purple</td>
<td>squiggle</td>
<td>3</td>
<td>outlined</td>
</tr>
</tbody>
</table>

A set is defined as exactly three cards where within each characteristic, the three cards are all the same or all different. The goal of the game is to find as many sets of cards as possible.

For example, consider the three cards containing:

2 red, striped ovals
2 purple, outlined ovals
2 green, solid ovals
In the category of Color, the three cards are all different. In the category of Shape, the three cards are all the same. In the category of Number, the three cards are all the same. In the category of Pattern, the three cards are all different.

Therefore, these three cards make a set

Now consider the three cards:

2 red, striped ovals
2 purple, outlined ovals
3 green, solid ovals

In the category of Color, the three cards are all different. In the category of Shape, the three cards are all the same. In the category of Number, two of the cards are the same and one is different. In the category of Pattern, the three cards are all different.

These three cards are NOT a set.

If you would like to see Sets, you can check out this website: http://www.setgame.com/instructions
You can also play online here: https://smart-games.org/en/set/multiplayer

a. You pick out three cards from the full deck (without replacing a card after you have picked it). What is the probability that the three cards that you pick make a set? Show your work.

b. What is the expected number of sets in the first 12 cards?

c. Given that the first 3 cards that you picked from the full deck are a set, what is the probability that the three cards all have different colors?

d. What is the probability that there is no set in the first 4 cards that you pick from the deck of 81? You can leave your answer as an expression if you choose.

Problem 10

Duncan know that \( p_1 \Rightarrow p_2 \) is logically equivalent to \( \neg p_1 \lor p_2 \). He is trying to figure out if this can be extended for any number of terms.

Use induction to help Duncan prove that for \( n \in \mathbb{Z}, n \geq 2 \),

\[
p_1 \Rightarrow (p_2 \Rightarrow (\ldots (p_{n-1} \Rightarrow p_n) \ldots)) = \neg p_1 \lor \neg p_2 \lor \cdots \lor \neg p_{n-1} \lor p_n.
\]

Note: The \ldots between the closing parenthesis just represent the sequence of closing parenthesis.