Midterm 1 Review Problems

Due: Never

All homeworks are due at 12:55 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

Suppose A and B are sets. For each of the following, either prove the statement using the element method, or give a counterexample.

a. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

b. $\mathcal{P}(A \setminus B) = \mathcal{P}(A) \setminus \mathcal{P}(B)$

c. $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$

Problem 2

Let S be a set with cardinality n. Prove using a bijection that the number of subsets of S of size k is equal to the number of subsets of size $n - k$.

Problem 3

a. Prove that provided two bijections $f : Q \to R$ and $g : R \to Q$, their composition, $f \circ g$, is a bijection for any non-empty sets Q, R. The composition is defined as $f \circ g(x) = f(g(x))$ for any $x \in R$.

b. Applying an arbitrary function repeatedly to some initial value is called an *iterated map*. Let f be a bijection $f : R \to R$.

   Let g_n be a function $g_n : R \to R$ that applies f n times where $n \geq 1$.

   That is, $g_n(x) = f(f(f(\ldots f(x)\ldots)))$ where f is applied n times.

   Prove that g_n is a bijection using induction.

Problem 4

Let R be an equivalence relation on A, where $|A| = n$.

a. What is the minimum size of R, in terms of n? Justify why you can achieve this size and why no smaller size can be achieved.

b. What is the maximum size of R, in terms of n? Justify why you can achieve this size and why no larger size can be achieved.

c. Suppose n is even. What can you say about the parity of $|R|$? Justify your response.

d. Suppose n is odd. What can you say about the parity of $|R|$? Justify your response.

Problem 5

Suppose you have a stack of n pancakes of different sizes on a plate and a spatula. Further suppose that you can place your spatula under any of the n pancakes and flip the stack of pancakes above your spatula upside-down. In other words, one flip will reverse the order of the pancakes from the top of the stack of pancakes to your spatula.

a. Describe an algorithm (procedure) to sort the n pancakes from largest at the bottom to smallest at the top using at most $2n$ flips.

b. Use induction to prove the correctness of your algorithm, namely that it will in fact sort the pancakes in at most $2n$ flips.

Problem 6

Prove that if $4 \mid n - 3$, then $8 \mid n^2 - 1$.

Problem 7

For $m \in \mathbb{Z}$, define the relation R_m on $M = \{1, \ldots, m-1\}$ by

$$\{(x, y) \mid \exists a, b \in \mathbb{Z}^+, \text{ such that } x^a \text{ and } y^b \text{ have the same remainder when divided by } m\}.$$

Prove that $\forall m \in \mathbb{Z}^+, R_m$ is an equivalence relation.

Problem 8

For $a, n \in \mathbb{Z}^+$, prove the following identity by induction on n:

$$(a+1)^n \equiv an + 1 \pmod{a^2} \tag{1}$$

Problem 9

We say integer a is divisible by integer b if there exists an integer m such that $a = bm$.

a. For a three-digit number ABC, where A, B, and C are the digits, prove that ABC is divisible by 7 if and only if $AB - (2 \cdot C)$ is divisible by 7. AB is a two-digit number with digits A and B.

   For example: The number 511 is divisible by 7 because $51 - 2(1) = 49$, which is divisible by 7.

b. Let's look at another type of integer that's divisible by 7. For digits A, B, and C, consider the function $\textsc{Repeat}(ABC) = ABCABC$, mapping three-digit integers to six-digit integers. For example, $\textsc{Repeat}(123) = 123123$ and $\textsc{Repeat}(442) = 442442$. (A digit is an integer between 0 and 9, inclusive.)

   Prove that for any three digits X, Y, and Z, $7 \mid \textsc{Repeat}(XYZ)$.

Problem 10

Let m be an integer with exactly three factors: 1, p, and m, where $1 < p < m$.

a. Prove that p is prime.

b. Prove that $m = p^2$.

c. Imagine a series of n doors (numbered 1 through n), each of which is initially closed. A line of n people (also numbered 1 through n) walk past the doors, opening or closing them in the following manner: person i toggles[1] every door numbered with a multiple of i. Prove that, once every person is done, door d is open if and only if d is a perfect square.

[1] That is, opens the door if it is closed, or closes it if it is open.