Recitation 3

Induction

Hello Induction

Why does induction work?

Let's consider an infinite ladder (the best kind of ladder). Suppose we can prove to you both of the following things:

a. You can get to the 1^{st} step of the ladder.

b. If you can get to the k^{th} step of the ladder, then you can get to step $k+1$.

$\star$ **TASK** $\star$ Why is it the case that for all $n \geq 1$, you can get to the n^{th} step of the ladder? Discuss with your neighbors.

Why are we talking about climbing infinite ladders? Well, it turns out this is a good way to think about how induction works.

The *base case* says that we can reach the first step of the ladder.

The *inductive hypothesis* says that we can get to the k^{th} step of the ladder.

The *inductive step* says that if we can get to the k^{th} step of the ladder, then we can get to step $k+1$.

Therefore, once we get to step 1, we can get to step 2. Once we get to step 2, we can get to step 3. And so on for all steps of the infinite ladder.

Induction Template

We will now review the template for an inductive proof.

For example, say we are trying to prove that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ is true for all $n \in \mathbb{N}$.[1]

1. Define the predicate $P(n)$. Recall that a predicate is a function that takes in an argument, n, and evaluates to true or false.

   Let $P(n)$ be the predicate that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

[1] If you're not familiar with this notation, check out `https://www.mathsisfun.com/algebra/sigma-notation.html`

2. Make the assertion that for all $n \geq a$, where a is the smallest value we are considering, $P(n)$ holds.

   For all $n \geq 0$, $P(n)$ holds.

3. Show that the base case is true. For some proofs, we may want multiple base cases.

   We will first show $P(0)$ is true. $\sum_{i=0}^{0} i = 0$ and $\frac{0(0+1)}{2} = 0$ so they are equal as needed.

4. State the inductive hypothesis. If you are using standard induction then you will assume $P(k)$ is true for some fixed, arbitrary integer $k \geq a$, where a is your base case value. If you are using *strong induction*, then you will assume $P(i)$ is true for all $i \leq k$. Sometimes, you may need multiple base cases, and you'll want k to be greater than or equal to the biggest of them. If you do need multiple base cases, you'll usually detect this when writing your inductive step (so you'll need to go back and add more base cases!). In this example though, it turns out we only need one.

   Assume $P(k)$ is true for some fixed, arbitrary integer $k \geq 0$.

5. Show that $P(k+1)$ is true given the inductive hypothesis.

   We will now show that $\sum_{i=0}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$.

   We know that $\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1)$.

   By our inductive hypothesis $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

   Therefore

$$\begin{aligned}\sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^{k} i\right) + (k+1)\\ &= \frac{k(k+1)}{2} + (k+1)\\ &= \frac{k(k+1) + 2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2}\\ &= \frac{(k+1)((k+1)+1)}{2}\end{aligned}$$

   as needed.

6. State that because the base case, $P(a)$ holds, and because $P(k) \implies P(k+1)$, we have that for all $n \geq a$, $P(n)$ holds.

   Because the base case, $P(0)$ holds, and because $P(k) \implies P(k+1)$, we have that for all $n \geq 0$, $P(n)$ holds. □

Warm-up

Prove by induction that for all $n \geq 2$,

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})...(1 - \frac{1}{n^2}) = \frac{n+1}{2n}$$

Let P(n) be the predicate that

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})...(1 - \frac{1}{n^2}) = \frac{n+1}{2n}$$

defined for integers $n \geq 2$.
Base Case: $1 - \frac{1}{2} = \frac{3}{4}$. $\frac{2+1}{4} = \frac{3}{4}$. Hence, P(2) holds.
Inductive Hypothesis: Suppose that for some fixed, arbitrary integer $k \geq 2$, P(k) holds. That is, $(1 - \frac{1}{2^2})...(1 - \frac{1}{k^2}) = \frac{k+1}{2k}$.
Inductive Step: Now, consider $(1 - \frac{1}{2^2})...(1 - \frac{1}{k^2})(1 - \frac{1}{(k+1)^2})$. By the I.H., we know that it is equal to $\frac{k+1}{2k}(1 - \frac{1}{(k+1)^2})$. But this is equal to $\frac{k+1}{2k} - \frac{k+1}{2k(k+1)^2}$. This is equal to $\frac{(k+1)^2-1}{2k(k+1)}$. This is equal to $\frac{k^2+2k}{2k(k+1)}$. This is finally equal to $\frac{k+2}{2(k+1)}$, as needed. Thus, P($k+1$) us true.
Conclusion: Since P(2) is true and for any integer $k \geq 2$, P(k) implies P($k+1$), P(n) is true for all integers $n \geq 2$

(!!!) Checkpoint - Call a TA over

Strong Induction

Consider a candy bar with n squares in a row. Suppose we want to break this candy bar up into n squares. How many breaks should we perform?

Figure 1: A Delicious Candy Bar

Claim: For all $n \geq 1$, any sequence of $n-1$ breaks will reduce a candy bar of n squares into single squares.

$\star$ **TASK** $\star$ Prove this claim by induction (Hint: you'll want to use *strong induction*).

Let P(n) be the predicate that given a bar of n squares, any sequence of $n-1$ breaks will reduce it to single squares.

Base Case: P(1): Given a bar with 1 square, 0 breaks leaves us with one square. Hurrah!

Inductive Hypothesis: Let k be some arbitrary positive integer. For all positive integers $i \leq k$, assume P(i). That is, assume that for a bar of length i, any sequence of $i-1$ breaks will reduce it to single squares.

Inductive Step: Consider a candy bar with $k+1$ squares. Make some break. Let's say we make it after the pth square, where p is some positive integer. Then, we have two candy bars, one of length p, the other of length $k+1-p$. Both p and $k+1-p$ are positive integers less than $k+1$, as we are breaking the bar into two positive pieces. By our inductive hypothesis, we know that it takes $p-1$ breaks to break up the first, and $(k+1)-p-1$ to break up the second. This is a total of: $(p-1)+((k+1-p)-1) = k+1-2$ breaks. We then have that our initial break plus this number of breaks is: $1+(k+1-2) = (k+1)-1$ breaks, as needed, so P(k+1) is true.
Conclusion: Thus, as P(1) is true and for any positive integer k, P(1)...P(k) imply P(k+1), P(n) is true for all positive integers n.

⋆ **TASK** ⋆ You are climbing a stair case, and you are able to step either one stair at a time or two stairs at a time. Show that the number of unique ways to climb to the n^{th} stair, where $n \geq 0$, is equal to the $n+1^{th}$ term in the Fibonacci sequence. Let the *ith* term, denoted as $F(i)$, of the Fibonacci sequence be defined as follows:

$$F(1) = 1$$
$$F(2) = 1$$
$$F(n) = F(n-1) + F(n-2)$$

Hence, the sequence looks like: 1, 1, 2, 3, 5...

Let P(n) be the predicate that the number of ways to get to the n^{th} stair is the $n+1^{th}$ term of the Fibonacci sequence.
Base Case: The number of ways to get to the 0^{th} stair (that is to go nowhere) is 1: stay where you are. $F(0+1) = F(1) = 1$, so P(0) holds.
The number of ways to get the first stair is also 1: you have to climb the stair. $F(1+1) = F(2) = 1$, so P(1) also holds.
Inductive Hypothesis: For some positive integer k, assume P(k-1) and P(k) hold. That is, assume the number of ways to get to the $k-1^{th}$ step is $F(k)$ and the number of ways to get to the k^{th} step is $F(k+1)$.
Inductive Step: If you are on the $k+1^{th}$ stair, there are two ways you could have gotten there: up one step from the k^{th} stair or up two steps from the $k-1^{th}$ stair. By the IH, the number of ways to get to the $k-1^{th}$ stair is $F(k)$ and the number of ways to get to the k^{th} stair is $F(k+1)$, so the number of ways to get to the $k+1$ stair is just the sum of these quantities, $F(k)+F(k+1) = F(k+2)$. Thus, the number of ways to get to the $k+1^{th}$ step is the $k+2 = (k+1)+1^{th}$ Fibonacci number, so P(k+1) is true.
Note: from the $k-1^{th}$ stair, we can also get to the $k+1^{th}$ stair by taking one step twice, but in doing so we pass the k^{th} stair, so this is already accounted for and we shouldn't double count it.
Conclusion: As P(0) and P(1) are true and for any integer $k \geq 1$ P(k-1), P(k) implies P(k+1), P(n) holds for all integers $n \geq 0$

(!!!) Checkpoint - Call a TA over

Induction Challenges

Challenge 1: We say that an infinite set S is countable if there exists a bijection from S to $\mathbb{N}$. You may assume that $\mathbb{N}\times\mathbb{N}$ is countable (i.e. you do not need to show that the base case in fact holds). Now, prove by induction that $\mathbb{N}^k$ (i.e. $\mathbb{N}\times\mathbb{N}\times...\times\mathbb{N}$, k times) is countable for all $k \geq 2$.

Let P(k) be the predicate that N^k is countable.
Base Case: As given, N^2 is countable, so P(2) is true.
Inductive Hypothesis: For some arbitrary integer $k \geq 2$, assume P(k). That is, assume N^k is countable.
Inductive Step: Consider $\mathbb{N}^{k+1}$. By the inductive hypothesis, $\mathbb{N}^k$ is countable so there exists a bijection $f_k : \mathbb{N}^k \to \mathbb{N}$. We can now make a bijection g_{k+1} from $\mathbb{N}^{k+1}$ to $\mathbb{N}\times\mathbb{N}$. $g_{k+1}((x_1,...,x_{k+1})) = (f_k(x_1,...,x_k), x_{k+1})$. As by our base case, we know there is a bijection $f_2 : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, we can create the full bijection $f_{k+1} : \mathbb{N}^{k+1} \to \mathbb{N}$ as $f_{k+1}((x_1,...,x_{k+1})) = f_2(g_{k-1}((x_1,...,x_{k+1}))) = f_2((f_k(x_1,...,x_k), x_{k+1}))$
Conclusion: As P(2) is true and for any integer $k \geq 2$ P(k) implies P(k+1), P(n) holds for all integers $n \geq 2$.

SUPER Challenge 2: n (where n is 2 or greater) dragons are sitting in a circle so that every dragon can see every other dragon.

Every dragon has green eyes. However, no dragon knows its own eye color. Additionally, the dragons cannot talk, so they cannot inform each other of the fact that they have green eyes.

On day 1, Professor Littman comes and tells the circle of dragons that at least one of them has green eyes.

On the day a dragon realizes it has green eyes, it turns into a human that night.

Prove that on the nth night, the n dragons will turn into humans.

Let P(n) be the predicate that n dragons will all turn into humans on the nth night.
Base Case: If there are 2 dragons, then it takes them two nights to figure out they both have green eyes. This is because consider one of the dragons, D. On the first day, D sees that the other dragon, D' has green eyes. D' does not know that they have green eyes, but they see that D does. Hence, neither D nor D' realizes on the first day that they have green eyes. n the second day though, D sees that the other dragon, D', did NOT realize that D' has green eyes. Hence, D knows that D has green eyes! D' also knows that D' has green eyes! So on the second night, they both turn into humans. Thus P(2) is true.
Inductive Hypothesis: Now for some arbitrary integer $k \geq 2$, assume P(k). That is, it takes k nights for k dragons to turn into humans.
Inductive Step: Consider the case where there is $k + 1$ dragons and consider what one individual dragon is thinking. This one dragon sees k dragons with green eyes. Therefore, if this dragon does not have green eyes, she would expect the other dragons to turn into humans after k nights since they can see her, by the IH. Since they don't, this dragon realizes that she also has green and turns into a dragon on night $k + 1$. Thus, P(k+1) is true.
Conclusion: As P(2) is true and for any integer $k \geq 2$ P(k) implies P(k+1), P(n) holds for all integers $n \geq 2$.

Just for Fun Challenge: More Set Theory

Before we say goodbye to set theory for good, there is an important question to consider. Before we can get to this question though, we'll need to go through a few things.

First, we say a set A is *well-defined* when there exists no x such that x must both be in A and not be in A. In other words, our definition of the set A has to be non-contradictory; it cannot call for some x to be both in and not in A.

Second, a *predicate* $p(x)$ is a function that takes in an argument, x, and evaluates to either true or false. For example, $p(x)$ could be "x is red". If x were, in fact, red, $p(x)$ would evaluate to true. If x were not red, $p(x)$ would evaluate to false.

We are now ready for our important question. Suppose we write $\{x \mid p(x)\}$. Is $\{x \mid p(x)\}$ necessarily a *well-defined* set?

Perhaps at first glance, this question seems to have an obvious answer, and that answer is yes. We've taught you that $\{x \mid p(x)\}$ is just set builder notation for the set containing all things x that satisfy $p(x)$.

But with a bit more thought, there are some things that are surprising. For example, let's say $p(x)$ were "x exists in the world". Then $A = \{x \mid p(x)\}$ would be the set of all things in the world. Interestingly, if A really is a well-defined set, then it would have to contain itself. That is, A would have to contain A. Why? Well, the set of all things that exist in the world, well, exists in the world itself!

Before the 20th century, most mathematicians were convinced that for any predicate, $\{x \mid p(x)\}$ described a well-defined set. However, at the dawn of the 20th century, a mathematician named Bertrand Russell came along and posed the following question.

Russell's question

Consider S, where $S = \{X \mid X \text{ is a set, and } X \text{ is not a member of } X\}$. For example, $A = \{1, A\}$ would not be in S. However, $C = \{1, 2\}$ would be in S. That is, S is the set of all sets that do not contain themselves.

We want to consider the question: is S a member of S? That is, is S a member of itself? Let's consider both possibilities.

a. **Possibility 1:** Suppose S is a member of S. From the definition of S, what can we conclude?

> If S is a member of S, then we know that by the definition of S, S is not a member of S. This is bad, so we must have that S is not a member of S. Except...

b. **Possibility 2:** Suppose S is not a member of S. From the definition of S, what can we conclude?

> If S is not a member of S, then by the definition of S, we should include S in S! That is, S is, in fact, a member of S! This is bad, so we must have that S is a member of S. Except we've already shown that we can't.

c. Is S well-defined? Recall what it means for a set to be well-defined: we cannot have an x such that x must both be in and not in S.

> S is not well-defined. If S is a member of S, S is not a member of S. And if S is not a member of S, then S is a member of S. In either case, we must have that S is both a member and not a member of S.

d. Now, knowing what we know, is it really the case that for any predicate $p(x)$, $\{x \mid p(x)\}$ is a *well-defined* set?

> Nope. Russell's set is a counterexample!

e. What does this all mean? Discuss with your neighbors.

Checkpoint - Call over a TA