Recitation 2
Relations, Functions, and Infinity

Part 1: Relations

Definitions

Defn 1: A *relation* R on the sets A and B is a subset of the Cartesian product $A \times B$. A relation R on the set A is a subset of the Cartesian product $A \times A$.

Defn 2: An *equivalence relation* is a relation that is reflexive, symmetric, and transitive.

Defn 3: A relation R on A is reflexive if $\forall a \in A$, $(a, a) \in R$.

Defn 4: A relation R on A is *symmetric* if $\forall (a, b) \in R$, $(b, a) \in R$. Another way to define this is that a relation is *not* symmetric if $\exists (a, b) \in R$ such that $(b, a) \notin R$.

Defn 5: A relation R on A is *transitive* if $\forall (a, b), (b, c) \in R$, $(a, c) \in R$. Another way to define this is that a relation is *not* transitive if $\exists (a, b), (b, c) \in R$ such that $(a, c) \notin R$.

Defn 6: Let R be an equivalence relation on A. Then the *equivalence class* of $a \in A$, denoted $[a]_R$, is $\{x \in A \mid (a, x) \in R\}$. That is, $[a]_R$ is all of the elements to which a is related.

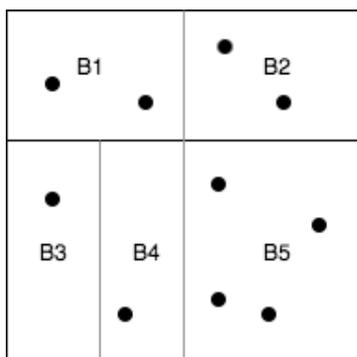## What is the point of an equivalence relation, anyway?

What does it mean for two things to be equal? It can depend on context. For example, you probably generally think of the numbers 2 and 4 as not being equal. However, maybe I want to consider the numbers 2 and 4 to be equal in some contexts because they are both even. We could be in a situation where we only want there to be two kinds of things: even things and odd things. We don't care about anything else like how big or how small the thing is.

An equivalence relation allows us to specify what things in the world are equal to each other, and what things aren't. This is why they're called **equivalence** relations!

An equivalence relation R splits up our world into categories, or equivalence classes. In a given equivalence class, all things within the class are things we consider equal, or equivalent, in the context of R.

We call the way the equivalence relation splits up our world a *partition*. A little more

formally, a *partition* of a set A is a list of subsets $B_1, \ldots, B_k$ of A s.t. every element of A is in some subset B_i, but no two subsets share an element.

One possible partition of some set A, where the dots in the square are distinct elements of A

Warm-Up

a. Consider the set $A = \{1, 2\}$.

    i. Write out the Cartesian product, $A \times A$.

           $\{(1,1), (1,2), (2,1), (2,2)\}$

    ii. Is $A \times A$ an equivalence relation on A?

           Yes

    iii. What is the equivalence relation on A with the smallest number of equivalence classes possible?

           $A \times A$, a world where everything is equal to everything.

iv. What is the equivalence relation on A with the largest number of equivalence classes possible?

> $\{(1, 1), (2, 2)\}$, where things are equal only if they are truly equal.

v. Is $R_0 = \{\}$ a relation on A?

> Yes, the empty set is a subset of every set

vi. Is R_0 symmetric? Why or why not?

> Yes, it does not violate our second definition of symmetric as there are no pairs.

vii. Is R_0 transitive? Why or why not?

> Yes, it does not violate our second definition of transitive as there are no pairs.

viii. R_0 is not an equivalence relation. Why?

> It is not reflexive.

b. Suppose R is an equivalence relation on B, and $R = \{\}$. What is B?

> The empty set, otherwise it is not reflexive.

Checkpoint - Call a TA over

c. Consider the set B of all students at Brown. For each of the following relations on B, determine whether they are reflexive, symmetric, transitive, or some combination of them. If it is an equivalence relation, then determine the equivalence classes of the relation.

   i. Two students are related if they are the same age.

   > Reflexive, symmetric, and transitive. Therefore equivalence relation. Equivalence classes are students of each age.

   ii. s_1 and s_2 are students and $(s_1, s_2) \in R$ if s_1 is younger than s_2.

   > Transitive but not reflexive or symmetric.

   iii. Two students are related if they are studying anthropology.

   > Symmetric and transitive but not reflexive.

   iv. Two students are related if they go to Brown.

   > Reflexive, symmetric, and transitive. Therefore equivalence relation. One equivalence class which consists of all students at Brown.

Checkpoint - Call a TA over

Part 2: Functions and Infinity

Definitions

Defn 1: A relation R on X and Y is a **function** if for every x in the domain X, x is mapped to one and only one y in Y, the codomain. Note that in the book this is called a *total function*, and function refers to a *partial function*, where for every x in the domain X, x is mapped to zero or one y in the codomain Y. In this class, we will use function to mean total function and partial function to mean partial function.

Defn 2: The **range** of a function f consists of all members of the codomain of f that are mapped to by some member of the domain of f.

Defn 3: $f : X \to Y$ is **injective** if for every $y \in Y$, there is *at most one* $x \in X$ such that $f(x) = y$. Equivalently if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. We can also define this property on all relations instead of just functions.

Defn 4: $f : X \to Y$ is **surjective** if for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$. We can also define this property on all relations instead of just functions.

Defn 5: $f : X \to Y$ is a **bijection** if it is both an injection and surjection.

Warm-Up 1

Let A be the set $\{1, 2, 3\}$. Consider the following relation on A, $R1 = \{(1, 2), (2, 1)\}$.

a. Is $R1$ a function?

> No; not all members of A are mapped to something in A. It is a partual function but not a total function.

Now consider $R2$, another relation from A to A: $\{(1, 2), (2, 1), (3, 2)\}$.

a. Is $R2$ a function?

> Yes.

b. If $R2$ is a function, what's its codomain? How about its range?

> The codomain is A. The range is $\{1, 2\}$.

Warm-Up 2

Using *potato* diagrams[1] explain the difference among the following relations:
1) a relation A on $\{0,1\} \times \{0,1\}$ that is not a function,
2) a relation B $\{0,1\} \times \{0,1\}$ that is a function but not injective, and
3) a relation C $\{0,1\} \times \{0,1\}$ that is both a function and injective.

> A will not include all members of the domain or will map one member of the domain to multiple values. B will include all members of the domain and will map each member to one and only one value in the codomain. However, at least two members will map to the same value in the codomain. C is just like B but this time, no two members of the domain will map to the same value in the codomain.

Warm-Up 3

For each of the following functions, determine if f is an injection, surjection, or neither. Also determine if it is a bijection.

Discuss your answers!

a. $f : \{0,1\} \to \mathbb{N}$

$f(0) = 1, f(1) = 0$

(Injective but not surjective.)

b. $f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = x^2$

(Not injective or surjective.)

c. $f :$ First Year Students $\to$ First Year Dorms

$f(\text{student}) = $ dorm that student lives in

(Not injective. Presumably surjective.)

d. $f :$ Brown University Students $\to$ Countries in the World

$f(\text{student}) = $ country where student is from

(Not injective, and sadly not surjective.)

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[1] A potato diagram is the very official name for a visualization of a relation R on A and B. These diagrams have two ovals, one for A and one for B, and they have an arrow from a to b in whenever (a, b) is in R.

e. $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x$

(Injective and surjective. Therefore bijective.)

f. *Challenge:* $f : \mathbb{R} \to \mathbb{R}$

$f(x) = \frac{x}{2}$

(Injective and surjective. Therefore bijective.)

Warm-up 4

a. If $f : X \to Y$ is injective, what can we say about the cardinalities of X and Y? Try making some diagrams where X has more elements than Y, fewer elements than Y, or the same number of elements as Y. When are you able to create an injection, and when are you not?

> If and only if $|X| \leq |Y|$, then there exists $f : X \to Y$ such that f is injective.

b. If $f : X \to Y$ is surjective, what can we say about the cardinalities of X and Y? Again, you might want to draw out some examples.

> If and only if $|X| \geq |Y|$, then there exists $f : X \to Y$ such that f is surjective.

c. Based on what you've found in the previous two questions, if $f : X \to Y$ is bijective, what can we say about the cardinalities of X and Y? When can we create a bijection between two sets, and when can we not?

> If and only if $|X| = |Y|$, then there exists $f : X \to Y$ such that f is bijective.

Checkpoint - Call a TA over

To Infinity and Beyond

Extending to the Infinite ∞

We previously worked mostly with finite sets. We're now going to shift focus to infinite ones, like the natural numbers. We're going to explore when we can say two infinite sets have the same cardinality and when we can say they don't, and we're going to think about what this all means.

When do two finite sets have the same cardinality? One answer is: when we can count the number of things in each set and determine they're the same. Another is: when there exists a bijection between them!

When do we say two infinite sets have the same cardinality? Unlike finite sets, we can't sit down and count each of the elements in an infinite set, but we can see if we can form a bijection between them! We say two infinite sets A and B have the same cardinality if there exists a bijection f between A and B.

Consider the following infinite sets:

- The natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$
- The even natural numbers $E = \{0, 2, 4, 6, 8, ...\}$
- The odd natural numbers $O = \{1, 3, 5, 7, 9, ...\}$

Do you think they all have the same cardinality?

Claim: $|E| = |O|$. There are as many even numbers as odd numbers.

Proof: We can prove it by giving a bijection $f : E \to O$, $f(x) = x + 1$.

Question: Do you think that $|E| = |\mathbb{N}|$? Why or why not?

$f(n) = 2n$

Challenge - Diagonalization

Let $\mathbb{N}$ denote the natural numbers and let $\mathbb{R}_{(0,1)}$ denote the real numbers between 0 and 1. We're going to show that there does not exist a bijection from $\mathbb{N}$ to $\mathbb{R}_{(0,1)}$. What this means is that even though $\mathbb{N}$ and $\mathbb{R}_{(0,1)}$ both have infinite cardinality, they are *different infinities*.

Task: It is true that we can form an injection from $\mathbb{N}$ to $\mathbb{R}_{(0,1)}$. Can you think of one such injection?

$f(x) = \frac{1}{1+x}$

Because of this, we know that the reason there cannot be bijection from $\mathbb{N}$ to $\mathbb{R}_{(0,1)}$ is that there is no injective mapping from $\mathbb{N}$ to $\mathbb{R}_{(0,1)}$ that is also *surjective*.

But let's assume for the sake of contradiction that we can form a bijection from $\mathbb{N}$ to $\mathbb{R}_{(0,1)}$. This implies that we can construct an injective and surjective function from $\mathbb{N}$ to $\mathbb{R}_{(0,1)}$. Let's try to do that now!

Let's call our mapping from $\mathbb{N}$ to $\mathbb{R}_{(0,1)}$, f. Now, consider the following table for an example f:

x	$f(x)$
0	0.42345...
1	0.23456...
2	0.87892...
3	0.48897...
4	0.78562...
$\vdots$	$\vdots$

Because f has an infinite domain, we can't write out the full table on the page. So let's just imagine we'd put *every single* natural number into the lefthand side of our table, and we'd paired each natural number with some arbitrary real number between 0 and 1. Additionally, to ensure f is injective, let's imagine we've paired each natural number with a different real between 0 and 1.

This is just one specific example function, though. We want to consider an *arbitrary* one! Here is a way to consider an arbitrary injection:

x	$f(x)$
0	$.a_{0,0}a_{0,1}a_{0,2}a_{0,3}a_{0,4}...$
1	$.a_{1,0}a_{1,1}a_{1,2}a_{1,3}a_{1,4}...$
2	$.a_{2,0}a_{2,1}a_{2,2}a_{2,3}a_{2,4}...$
3	$.a_{3,0}a_{3,1}a_{3,2}a_{3,3}a_{3,4}...$
4	$.a_{4,0}a_{4,1}a_{4,2}a_{4,3}a_{4,4}...$
$\vdots$	$\vdots$

Here, we have that $a_{i,j}$ represents the digit at the ith row and jth column of the rightside of our table. Each $a_{i,j}$ is some digit in $\{0,1,2,3,4,5,6,7,8,9\}$.

For f to be surjective, we know that it cannot be the case that there exists a real number between 0 and 1 that is not on the righthand side of our table. Take a few moments to convince yourself this is what surjectivity means.

But we're now going to try to come up with some real number between 0 and 1 that is necessarily not on the righthand side of our table. In other words, we will show our function is not surjective.

Task: Call over a TA to discuss the setup we just walked through.

Task: Provide some a real number between 0 and 1 such that it is not equal to any $a_{i,0}a_{i,1}a_{i,2}a_{i,3}a_{i,4}...$. Explain why this real number necessarily does not appear on the righthand side of our table.

Hint: You'll want to consider the following number: $.a_{0,0}a_{1,1}a_{2,2}a_{3,3}a_{4,4}...$. This number corresponds to the **diagonal** through all of the entries on the righthand side of our table. How can you construct some number, using this diagonal, that is necessarily not on the righthand side of the table? Think about how if for any integer n $f(n)$ were this number, it would be true that the nth position equaled $a_{n,n}$.

> Change $a_{i,i}$ in diagonal to get $a'_{i,i}$. If some $a_{i,i}$ is 1, set $a'_{i,i}$ to 2. If some $a_{i,i}$ is 2, set $a'_{i,i}$ to 1. Consider this new real number, $.a'_{1,1}a'_{2,2}...a'_{i,i}...$. We will show that this real number cannot be on the righthand side of our table. Can it be in the first row of the righthand side (i.e. the row that corresponds to 0)? No, because $a_{1,1}$ differs from $a'_{1,1}$. By the same argument, this new real number cannot be in any row. So we've now proved that some real number between 0 and 1 is necessarily never on the righthand side of the table.

This means that the set of all natural numbers has a smaller cardinality than the set of all real numbers between 0 and 1. This shows there are different infinite cardinalities. The integers and all sets with the same cardinality are called countably infinite, while the reals and all sets with the same cardinality (we could prove that $R_{(0,1)}$ is one such set!) are called uncountably infinite.