Recitation 1

Sets and Proofs

What's recitation?

Recitation is a space for you to work with other members of the CS22 community on problems that we hope will help you hone your understanding of the course material, get better at communicating with other folks about mathematical ideas, and practice for the homework. You'll also get to know the some of the TAs and ask them any questions about the course material that you're passionate about, barring specific questions about the homework for the week. This is **not** a space for you to do your homework.

Review

Defn 1: A **set** is a collection of objects with no repetition or order.

Defn 2: B is a **subset** of A if every element in B is also in A. This is written as $B \subseteq A$.

Defn 3: The **integers** are the set $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. The **non-negative integers** (also called the natural numbers) are the set $\mathbb{N} = \{0, 1, 2, ...\}$..

Defn 4: A number n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. A number n is **odd** if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Defn 5: A number n is **rational** if $n = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, where $b \neq 0$.

Defn 6: $\mathcal{P}(A)$, called the **power set** of A, is the set of all subsets of A.

Defn 7: $A \cup B$ denotes the **union** of sets A and B. This contains all the elements from A, and all of the elements from B.

Defn 8: $A \cap B$ denotes the **intersection** of sets A and B. This contains only the elements that appear in both of the sets.

Defn 9: $A - B$ denotes the **difference** of sets A and B. This contains elements that appear in A but not B.

Defn 10: $\overline{A}$ denotes the **complement** of A relative to some universal set U. $\bar{A} = U - A$, that is, it is everything except what is in A.

Defn 11: $|A|$ denotes the **cardinality** of A, which is a count of the number of elements contained in A.

Warm Up

Answer true or false to the following problems. Discuss your answers!

a. A is an arbitrary set. Answer true only if the statement is always true. That is, answer true only if for any possible set A, the statement is true.

   i. $A \subseteq A$ (T)
   ii. $\{\} \subseteq A$(T)
   iii. $\{\} \in A$ (F)
   iv. $B = \{A\}$. B is a set. (T)
   v. $C = \{A, A\}$. C is a distinct set from B. (F)

b. A is a set, and $\mathcal{P}(A)$ is the set of all subsets of A. Answer true only if the statement is always true.

   i. $A \in \mathcal{P}(A)$ (T)
   ii. $A \subseteq \mathcal{P}(A)$ (F)
   iii. $\emptyset \in \mathcal{P}(A)$ (T)
   iv. $\emptyset \subseteq \mathcal{P}(A)$ (T)
   v. $\{A, \emptyset\} \subseteq \mathcal{P}(A)$ (T)

c. If $A = \{1, 2, 4\}$ then $\{2, 4\} \in \mathcal{P}(A)$ (T)

d. $\mathbb{N} \subseteq \mathbb{Z}$ (T)

e. $\{0, 1, 9\} \subseteq \mathbb{N}$ (T)

f. $\{-1.5, 9\} \subseteq \mathbb{Z}$ (F)

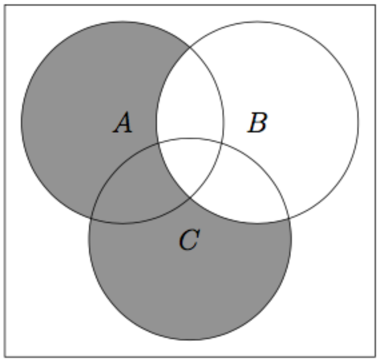
g. S is the set of students in CS22. B is the set of students at Brown. Duncan is a student in CS22.

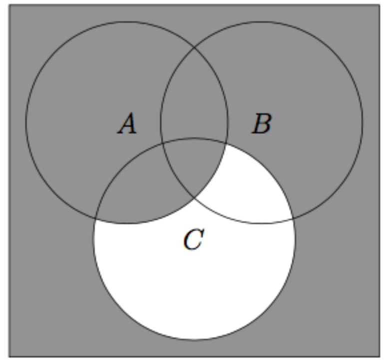
   i. $S \subseteq B$ (T)
   ii. Duncan $\subseteq S$ (F)
   iii. Duncan $\in S$ (T)
   iv. $\{\text{Duncan}\} \subseteq B$ (T)

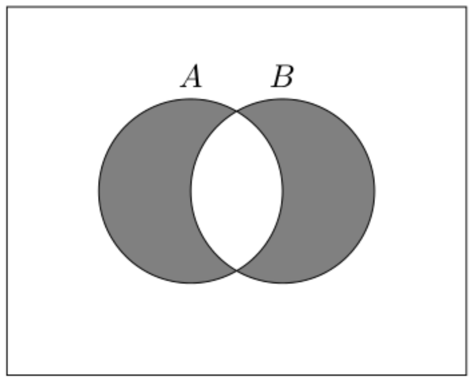
h. Let $\mathbb{Q}$ be the set of rational numbers.

   i. $\mathbb{Q} \cap \mathbb{N} = \mathbb{N}$ (T)
   ii. $\mathbb{Q} \cup \mathbb{N} = \mathbb{R}$ (F)

i. In each of the following Venn diagrams, A, B, and C are sets and are assumed to be subsets of a universal set (denoted by the rectangle). Write an set algebraic expression (i.e. one involving union, intersection, difference, and complement) in terms of A, B, and C for the shaded in region.







Solutions:

$(A \cup C) - B$

$\overline{C} \cup A$

$(A \cup B) - (A \cap B)$

j. *Challenge:* Call a the cardinality of A and b the cardinality of B. Call s the cardinality of $A \cap B$. For the third picture, what is the cardinality of the set formed from the expression you derived?

Solution: $a + b - 2s$

Checkpoint - Call a TA over.

Proof Techniques

Direct Proof

A direct proof is one that begins with statements you know to be true, make logical jumps to new statements, and eventually ends up with what you are trying to prove.

Here is an example.

Claim: If n is odd, then n^2 is odd.

Proof (direct): We know that n is odd, so $n = 2k + 1$ for some $k \in \mathbb{Z}$.

So $n^2 = (2k+1)(2k+1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$,

where $m = 2k^2 + 2k$.

Since m is an integer, n^2 is odd. $\square$

a. Prove that the product of an even number and odd number is even.

> Consider n odd and m even. $n = 2a + 1$ and $m = 2b$.
>
> Their product is $(2a+1)(2b) = 4ab + 2b = 2(2ab + b)$.
>
> Since $2ab + b$ is an integer their product is even.

b. Prove that the product of two rational numbers is rational.

> Consider $n = \frac{a}{b}$ and $m = \frac{c}{d}$.
>
> Their product is $nm = \frac{ac}{bd}$. Since the product of two integers is an integer, we have just expressed nm as the ratio of two integers. As b and d are both nonzero, so is their product. Therefore nm is rational.

Negation and Counterexample

Sometimes, we don't ask you to prove claims are true; we instead ask you to prove that they are false! This means your task is to prove the *negation* of the claim.

What is the negation of a claim? Let's think about it.

Question: Suppose Phoebe says to Jay: "Everyone in the world likes PB and J!". What would Jay need to show Phoebe to convince Phoebe this was false? Circle your choice.

1. Every person in the world hates PB and J.

2. There is at least one person in the world who does not like PB and J.

3. There are at least 22 people in the world who do not like PB and J.

Answer: 2

Question: Now, practice negating each of the following claims. Write the negated version of the claim next to the original.

1. All CS22 students like potatoes.

2. There exists a student in CS22 who is a PB and J fanatic.

3. $\forall x \in \mathbb{Z}$, if x is even, $2x$ is odd.

4. For an arbitrary $x \in \mathbb{Z}$, if x is even, $2x$ is odd.

5. For some x, if x is odd, $x + 1$ is odd.

Answer Key:

1. There exists a student in CS22 who does not like potatoes.

2. All students in CS22 are not PB and J fanatics. **Be sure to note the different between NOT all students are and ALL students are not.**

3. There exists an $x \in \mathbb{Z}$ such that if x is even, $2x$ is not odd.

4. There exists an $x \in \mathbb{Z}$ such that if x is even, $2x$ is not odd.

5. For all x, if x is odd, $x + 1$ is even.

Question: Try to phrase "All CS22 students like potatoes" in three or more different ways.

Use all, for all, arbitrary, any, etc.

Sometimes, we can show that a claim is false (i.e. the negation of the claim is true) by providing a **counterexample**. For example, suppose Phoebe makes the claim that if xy is rational then x and y are rational.

Jay can disprove Phoebe's claim by coming up with a counterexample. For example, if $x = \sqrt{2}$ and $y = \sqrt{2}$, then $xy = 2$, which is rational.

However, you **cannot** prove a claim by showing one example of it. We cannot prove Phoebe's claim to be true by just providing a specific x that is rational and a specific y

that is rational such that xy is rational. Also, just because we find a counterexample to a claim does not mean that we've ruled out the existence of some specific x that is rational and some specific y that is rational such that xy is rational.

Question: Suppose each of the claims from before were false. Determine which of them you could prove were false by providing a counterexample and which you could not. What do you notice?

1. All CS22 students like potatoes.
2. There exists a student in CS22 who is a PB and J fanatic.
3. $\forall x \in \mathbb{Z}$, if x is even, $2x$ is odd.
4. For an arbitrary $x \in \mathbb{Z}$, if x is even, $2x$ is odd.
5. For some x, if x is odd, $x + 1$ is odd.

Answer Key: *The moral of the story is:* we can disprove for all claims with counterexamples. We cannot disprove there exists statements with counterexamples.

1. Can provide counterexample
2. Cannot provide counterexample
3. Can provide counterexample
4. Can provide counterexample
5. Cannot provide counterexample

Question: Now, disprove the following statement by providing a counterexample. If xyz is rational, then x, y, and z are rational.

Consider $x = \sqrt{2}$, $y = \sqrt{3}$, $z = \sqrt{6}$. $xyz = 6$ which is rational.

Checkpoint - Call a TA over.

Proof by Contradiction

Let's call "math" the set of all statements about the mathematical world that we can prove are true. Consider a statement x in math. Can the negation of x also be in math? No! That is, we can't have that a claim and its negation are both true. This would be a **contradiction**.

For example, it cannot be the case that 3 is odd and 3 is not odd. Only one of these statements can be true, and one of them has to be true.

This idea motivates a proof technique called *proof by contradiction*. The general idea is this.

Say we have some statement T that we are trying to prove is true.

To prove T is true by contradiction:

1. We begin by assuming T is NOT true. That is, we assume that the negation of T is true.

2. Assuming T is not true leads us to a contradiction. That is, by making logical leaps from T being true, we arrive at the fact that a statement x and its negation both have to be true.

3. But x and its negation cannot both be true; this is a contradiction. We got to this contradiction by assuming T was false. Therefore, we know T cannot be false; i.e. T is true.

Here is an example.

Claim: $\mathbb{N}$ is an infinite set.

Proof: Assume for sake of contradiction that there are a finite number of natural numbers. Then there must be a largest natural number. Say this largest number is m.

However, $m + 1$ is still a natural number, and $m + 1$ is larger then m.

This is a contradiction to the fact that m is the largest natural number.

Assuming $\mathbb{N}$ was finite led to a contradiction, and therefore $\mathbb{N}$ is infinite. $\square$

Your turn. Prove the following statements by contradiction.

a. 2 is an even number.

> Assume for sake of contradiction that 2 was odd. Then $2 = 2k + 1$ for some integer k. Then $k = \frac{1}{2}$ which is a contradiction since k is an integer.

b. Suppose $a, b \in R$. If a is rational and ab is irrational, then b is irrational.

> Since a is rational, $a = \frac{x}{y}$ for some $x, y \in \mathbb{Z}, y \neq 0$. Assume for sake of contradiction that b is rational. Then $b = \frac{z}{w}$ for some $z, w \in \mathbb{Z}, w \neq 0$. This means $ab = \frac{xz}{yw}$ and $xz, yw \in \mathbb{Z}, yw \neq 0$, so ab is rational, which is a contradiction.

c. If A and B are sets, then $A \cap (B - A) = \emptyset$.

> Assume for the sake of contradiction that $A \cap (B - A) \neq \emptyset$, that is, there exists some x such that $x \in A \cap (B - A)$. By the definition of intersection, this implies $x \in A$ and $x \in B - A$. But, by the definition of set difference, $B - A$ means $x \notin A$, which is a contradiction.

d. *Challenge:* Consider "the smallest positive integer not definable in fewer than twelve words". Show that this integer cannot exist.

> Say there was some smallest positive integer not definable in fewer than twelve words. Call this integer s. We could then define s in fewer than twelve words by saying "the smallest positive integer not definable in fewer than twelve words." This is a contradiction.

Checkpoint - Call a TA over.