Homework 6

Due: Wednesday, April 1

All homeworks are due at 12:55 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

a. Duncan has developed insomnia wondering about the following: Can we express the OR operation using only IMPLIES operations? Justify your answer and help Duncan sleep again.

b. Suppose we define a new operation $\star$ on logical propositions such that

$$x \star y \equiv \text{NOT } (x \text{ AND } y)$$

Create a truth table for each of the following expressions, and state which logical operator the expression is equivalent to.

i $x \star x$

ii $(x \star y) \star (x \star y)$

iii $(x \star x) \star (y \star y)$

iv $(x \star (x \star y)) \star (y \star (y \star x))$

Problem 2

a. Suppose p and q are propositions such that p IMPLIES q is False. Determine the truth values of:

(i) (NOT p) IMPLIES q

(ii) p OR q

(iii) q IMPLIES p

b. You are now going to design a circuit that takes as input two 1-bit binary numbers A and B and outputs whether or not $A > B$, $A < B$, or $A = B$. Namely, the circuit should have two inputs, the bits A and B, and three outputs G, E, and L. For any input, exactly one of G (greater), E (equal), or L (less) should be 1. Note that G corresponds to $A > B$.

(i) Write out a truth table equivalent to this circuit.

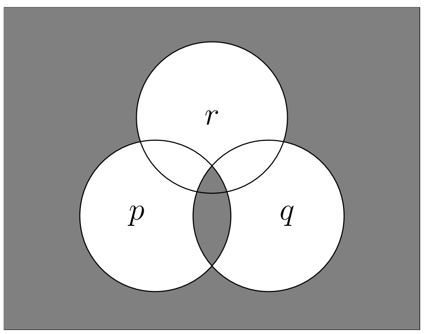
(ii) Draw your circuit. Please use only AND, OR, and NOT gates with at most two inputs per gate. Note that we care only about the correctness of the circuit, not its complexity.

Be sure to explain how your circuit works.

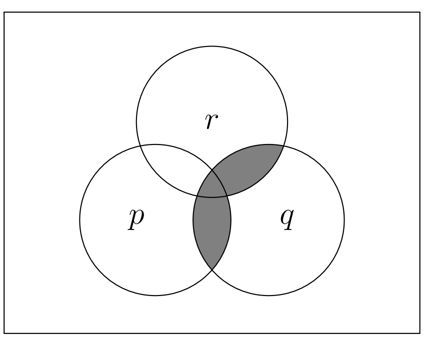
Problem 3

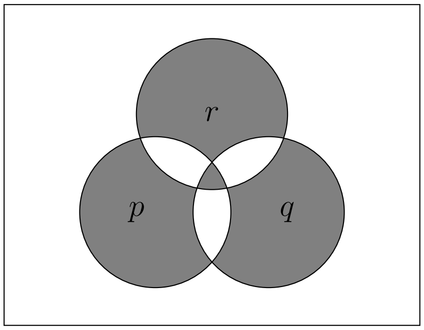
A common way to visually represent boolean formulas is to use Venn diagrams. In a Venn diagram, each circle corresponds to a proposition. The proposition is true for the area inside the circle and false outside of it. A space is shaded if that combination of propositions causes the formula to evaluate to true; it is unshaded if it evaluates to false.

For example, (p AND q) OR (NOT p AND NOT q AND NOT r) is equivalent to the following Venn diagram:



i. For each Venn diagram, find an equivalent boolean formula.

a. 

b. 

ii. For each boolean formula, draw a Venn diagram that corresponds to it. Venn diagrams may be drawn by hand or in Latex. (If you use Latex, `http://users.ju.edu/hduong/math220/venn_diagrams.pdf` explains how to use to tikz package to draw Venn diagrams.)

a. (p AND q) OR (p AND r) OR (q AND r)
b. (p IMPLIES q) XOR r

Problem 4

For each of the following expressions, convert it to CNF. Using truth tables or logical rewrite rules, prove your converted expression is logically equivalent to the given one.

a. Expression: (p AND NOT q) OR NOT p
b. Expression: (p AND NOT q) OR (NOT p AND q)
c. Expression: NOT (p AND (q OR r))
d. Expression: (p IMPLIES q) IMPLIES r

Problem 5

Let B be the set of all possible logical expressions. Define the relation R over B such that, for $a, b \in B$, $a\ R\ b$ if and only if a and b return the same output given the same input.

The equivalence class of a logical expression x is $[x]_R = \{a \in B \mid a\ R\ x\}$. That is, an equivalence class of a proposition is the set of logical expressions that for every possible input, give the same output.

Construct a bijection between equivalence classes of three-input, one-output logical expressions and subsets of a set containing eight items. Be sure to prove your function is indeed a bijection.