Homework 5

Due: Wednesday, March 4

All homeworks are due at 12:55 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

Crisp and Creme are playing a *very fun* game. The game starts with two distinct, positive integers a and b written on a blackboard. On a given player's turn, they write a new positive integer on the board that is the difference of two integers already present on the board. If they cannot do so, they lose.

For example: suppose that 12 and 15 are on the board initially. Crisp plays first and writes 3, which is $15 - 12$. Then Creme writes $9 = 12 - 3$. Then Crisp plays $6 = 15 - 9$. Creme cannot write any new integers, so Creme loses.

a. Prove that every number on the board at the end of the game is a multiple of $\gcd(a, b)$.

b. Prove that every positive multiple of $\gcd(a, b)$ up to $\max(a, b)$ is on the board at the end of the game.

c. Given that Crisp can choose to go first or second, describe a strategy that allows Crisp to win every time. Explain why your strategy is correct.

Problem 2

a. Use induction to show that $4^{n+1} + 5^{2n-1}$ is divisible by 21 whenever n is a positive integer.

b. Use induction to show that $4|(3^{2k-1} + 1)$ for any positive integer k.

c. Prove that $\frac{21n+4}{14n+3}$ is irreducible for any positive integer n. This means that the numerator and the denominator have no common factors.

Hint: consider the greatest common divisor of the numerator and denominator. You can hand simulate Euclid's algorithm to compute it

Problem 3

Split the numbers from one to seven into two sets, A and B, with $|A| = 4$ and $|B| = 3$. Add the product of the numbers in A to the product of the numbers in B. There's exactly one way to split up the numbers so the resulting sum is prime. What is the prime? How did you come to your conclusion? You do not need to prove your final result is prime. Do not use brute force.

Hint: Think of what happens if there's an $a \in A$ and $b \in B$ such that $\gcd(a, b) > 1$

Problem 4

In each of the following, find a value of x that satisfies the given congruence, or argue that no such x exists. Show how you found x.

a. $4(x - 3) \equiv 8x - 3 \pmod{41}$

b. $7(x + 5) \equiv 2x \pmod{13}$

c. $6(x - 3) \equiv 5 \pmod{15}$

d. $3x + 4 \equiv 2(x - 5) \pmod{7}$

Problem 5

For each of the following, find a multiplicative inverse for the given element in two ways: first by using the Euclidean Algorithm, and second by using Euler's Theorem. If no inverse exists, state so and explain why.

a. $4 \pmod{17}$

b. $25 \pmod{21}$

c. $38 \pmod{19}$

d. $42 \pmod{33}$

e. $31 \pmod{23}$