Homework 3

Due: Wednesday, February 19

All homeworks are due at 12:55 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

Let A be a set with n elements. Let T be the set of all ordered pairs (X, Y) where X and Y are subsets of A. Let S be the set of 0/1/2/3 strings of length n. That is, elements of S are strings of length n where each character is 0, 1, 2, or 3. Give (and prove) a bijection between T and S.

Conclude that T and S must be the same size.

Problem 2

Use mathematical induction to prove the following generalization of one of DeMorgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, ..., A_n$ are subsets of a universal set U and $n \geq 2$.

Note: You can use the form of DeMorgan's Law presented in lecture without proof.

Problem 3

a. Prove by induction that for all positive integers n, there exists a positive integer m such that:

$$m^2 \leq n < (m+1)^2$$

b. Prove by contradiction that there exists a **unique** such m.

Problem 4

A relation R is antisymmetric if the following condition holds: if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

A *partial order* is a relation that is reflexive, antisymmetric, and transitive.

Prove that the divisibility relation $\mathcal{R} = \{(a, b) \mid \ a \text{ divides } b\}$ on the positive integers is a partial order.

Problem 5

The game of *Mini-nim* is defined as follows: Some positive number of donuts are placed on the ground. Two players take turns eating one, two, or three donuts. The player to eat the last donut loses.

Prove that the second player has a winning strategy if the number of donuts equals $4k + 1$ for integer $k \geq 0$.