Homework 2

Due: Wednesday, February 12

All homeworks are due at 12:55 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

a. Prove or disprove that a relation that is reflexive and symmetric is necessarily transitive.

b. Prove or disprove that a relation that is reflexive and transitive is necessarily symmetric.

c. Prove or disprove that a relation that is symmetric and transitive is necessarily reflexive.

Problem 2

Determine whether or not each of the following relations is an equivalence relation. Be sure to justify your answers.

a. The relation R on $\mathbb{Z}$ defined by the set of ordered pairs:

$$\{(a, b) \mid \quad |a - b| \leq 2\}.$$

b. The relation on $\mathbb{R}^2$ defined by the set of ordered pairs:

$$\{(a, b) \mid ||a|| = ||b||\},$$

where $||a||$ is the distance from a to the origin in $\mathbb{R}^2$ ($\mathbb{R}^2$ is the set of ordered pairs (x, y) where $x, y \in \mathbb{R}$, also known as the set of points in the plane, and the distance from a point (x, y) to the origin is defined as $\sqrt{x^2 + y^2}$.)

In addition to your proof, answer the following: given a fixed point $p \in \mathbb{R}^2$, the collection of all points related to p gives what familiar geometric object? That is, what is $\{x | x \; R \; p\}$?

c. Let $S = \{a, b, c, d\}$. Let R be the relation on S with the graph:

$$\{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c), (b, a), (c, b), (c, a)\}$$

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Problem 3

Determine whether each of the following statements is true or false, and explain why.

a. The powerset of the empty set has the empty set as a member.
b. The empty set is an element of every set.
c. The empty set is a subset of any set that does not have the empty set as a member.
d. Any set that has the empty set as a member must be the empty set.
e. The set containing the set containing the set containing the empty set has a cardinality of zero (here, A contains B means $B \in A$).
f. The set of all empty sets is the same as the powerset of the empty set.

Problem 4

a. i. Find all relations from $\{0, 1\}$ to $\{1\}$.
   ii. Find all relations from $\{1\}$ to $\{0, 1\}$.
b. i. Find all functions from $\{0, 1\}$ to $\{1\}$.
   ii. Find all functions from $\{1\}$ to $\{0, 1\}$.
c. Let $S = \{0, 1\}$, $T = \{t | t \subseteq S \times S\}$, and R be the set of all possible functions from S to S.
   i. Can an injection from T to R exist? If so, give one such injection and prove that this mapping is indeed injective. If not, prove why such a mapping cannot exist.
   ii. Can a surjection from T to R exist? If so, give one such surjection and prove that this mapping is indeed surjective. If not, prove why such a mapping cannot exist.
   iii. Can a bijection from T to R exist? If so, why? If not, why not?

Problem 5

Drayton Martin is the Vice President of Brand Stewardship at Dunkin' Donuts [1]. She is responsible for deciding which stores carry which flavors.

Consider the nonempty set of flavors $F = \{f_1, f_2, \ldots, f_n\}$ that a store can carry.

[1] https://www.linkedin.com/in/drayton-martin-a616413

Drayton can tell a store to carry any subset of these flavors (even the empty set). She decides to assign each of these subsets to exactly one store.

Prove that the number of stores that sell an even number of flavors is equal to the number of stores that sell an odd number of flavors. Prove this by constructing a bijection between the two set of stores.

Be sure to prove that your function is in fact bijective.