Homework 1

Due: Wednesday, February 5

All homeworks are due at 12:55 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

Prove the following claim via proof by contradiction:

For all real numbers x and y, if x is irrational and y is rational, then $x-y$ is irrational.

Problem 2

Given the following sets:

$$\begin{aligned} A &= \{a, b, c, d, g, f\} \\ B &= \{a, e\} \\ C &= \{b, d, e, g\} \\ D &= \{a, g\} \end{aligned}$$

a. Compute $(A \cap C) \cup D$

b. Compute $B \times ((A \cap C) \cup D)$

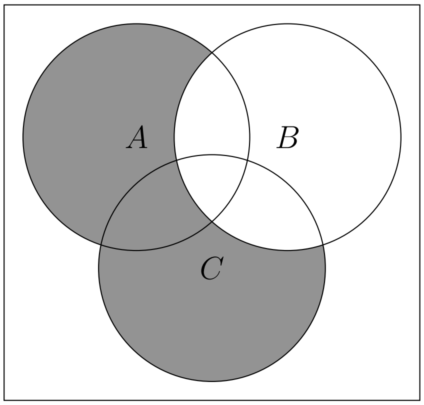
c. Compute $|\mathcal{P}(B \times ((A \cap C) \cup D))|$

d. How many distinct binary relations on $A \cap C$ are there? You need not justify your answer.

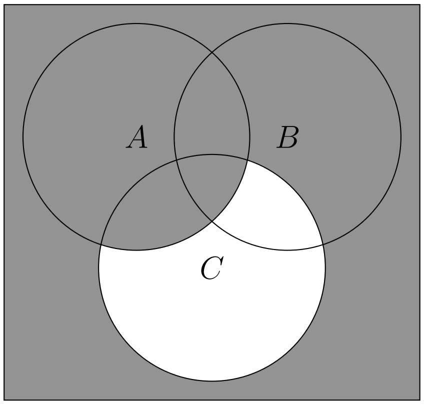
Problem 3

In each of the following Venn diagrams, A, B, and C are sets and are assumed to be subsets of a universal set (denoted by the rectangle). Write a set algebraic expression in terms of A, B, and C for the shaded region. (Try to keep your expression as simple as possible.)

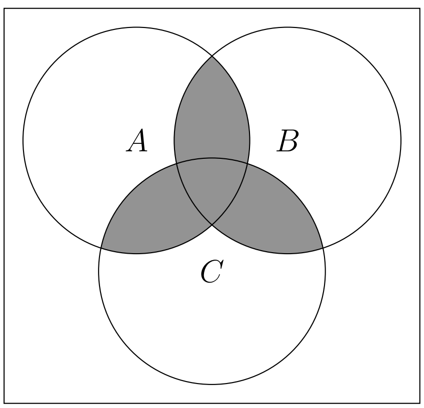
a.



b.



c.



Problem 4

a. Prove (using the "set element" method) or disprove the following claim:
   For any two sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.
b. Prove (using the "set element" method) or disprove the following claim:
   For any two sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Problem 5

a. A binary operation, $\star$, is an operation on a set that takes in two elements from the set and returns a third. For example, the addition operation $+$ is a binary operation on the integers.

We say that a binary operation $\star$ on a set S is *commutative* if

$$x \star y = y \star x \text{ for all } x, y \in S.$$

Let X be a finite set. For each of the following operations, prove whether or not the operation is commutative over $\mathcal{P}(X)$.

i Set union

ii Set intersection

iii Set difference

iv Symmetric difference

(Recall that the symmetric difference is defined as $(A \cup B) - (A \cap B)$. See the lecture notes and/or text for definitions of any other operations.)

b. Consider a binary operation $\star$ on a set S. An *identity element* for $\star$ is any $e \in S$ such that $e \star x = x \star e = x$ for all $x \in S$. For example, 0 is an identity element for the operation $+$ over the integers, because $x + 0 = 0 + x = x$ for all $x \in \mathbb{Z}$.

Let X be a finite set. Which elements in $\mathcal{P}(X)$ are identity elements for the operation $\cup$? Which elements in $\mathcal{P}(X)$ are identity elements for the operation $\cap$? Prove your response (note: this means you must both show why some elements are identity elements and why all others aren't).