C.S. 252 Prof. Roberto Tamassia
Computational Geometry Sem. II, 1992–1993

Planar Upward Drawings

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1 Theorem:

G is acyclic if and only if it has upward drawing.

Proof:

1. An upward drawing is acyclic.

   Follow the chain from the source, since it is upward then the Y coordinate will only increase; therefore it can never create a cycle. Note that the drawing need not be planar.

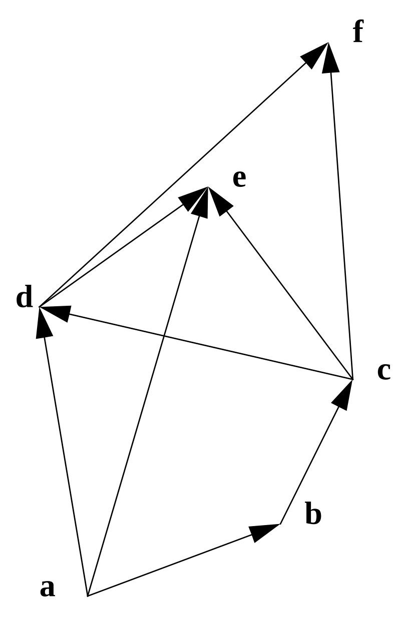

Figure 1: An upward drawing.

2. An acyclic graph G has an upward drawing.

   One way to show the existance of an upward drawing for G, would be to draw all the vertices of G along a vertical line, with source vertices having smaller Y value, which would make the graph upward. The algorithm would be as follows:

   – Find the sources (vertices with no incomming edge), set it's Y value.

– Do a DFS search to find the next vertices (after removing the sources) and assign the next Y value to it.

Although this drawing is not a very nice drawing of G, but indeed it is upward and the proof is complete.

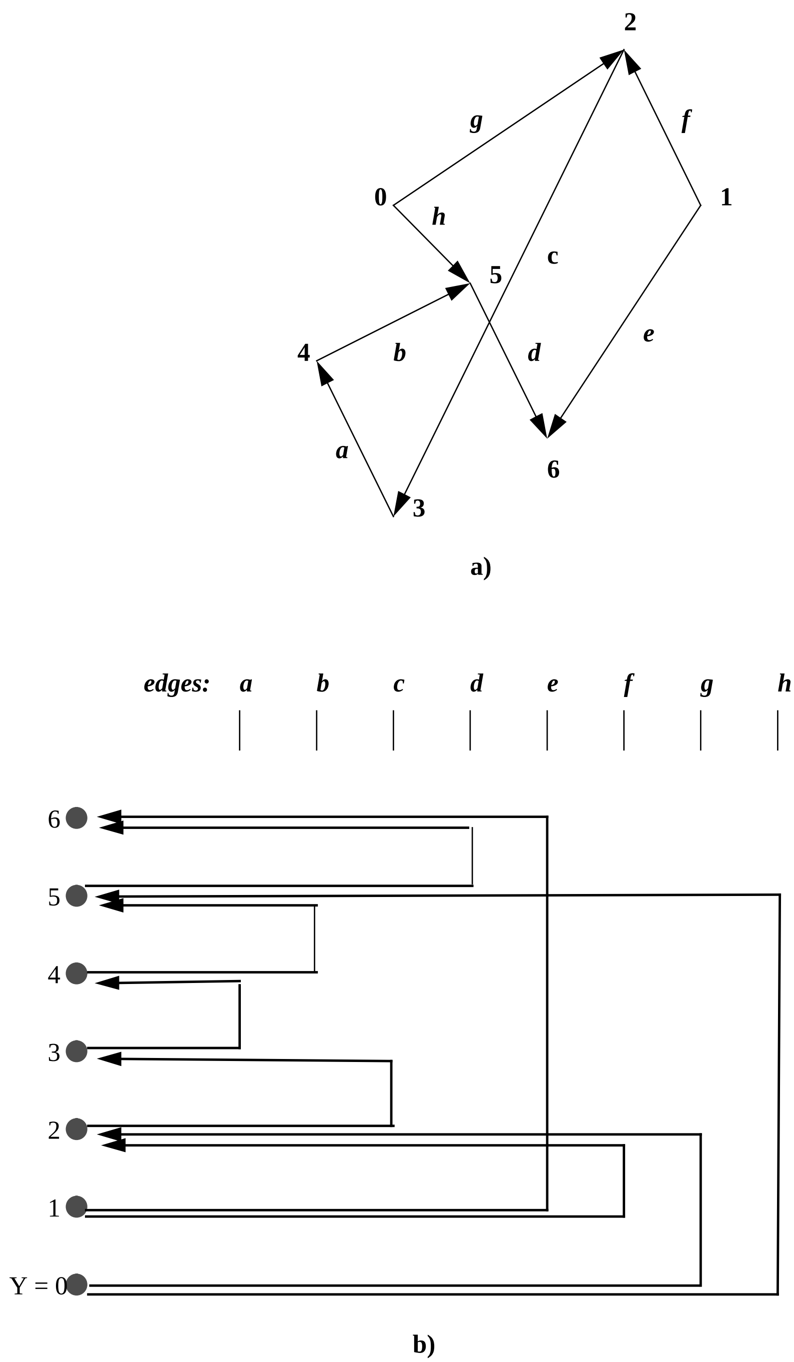

Figure 2: a) an acyclic drawing, b) it's upward drawing.

2 Upward Planarity Testing

Not every planar DAG admits a planar upward drawing. There are three different algorithms for testing the existence of an upward planar drawing of a digraph, however their complexity can be improved.

1. Di Battista and Tamassia's algorithm which proves the digragh is upward planar if and only if it is a subgraph of a planar st-digraph. St-digraphs have the following characteristics:

   — acyclic

   — planar

   — one source (s)

   — one sink (t)

   — with edge (s,t)

2. Hutton and Lubiw's $O(n^2)$ time complexity upward planarity testing for single-source digraphs. A recent research has improved the time complexity to $O(n)$.

3. Bertolazzi, Di Battista, Liotta and Mannino have designed an $O(n^2)$ time algorithm which tests upward planarity of digraphs with fixed embedding. this algorithm would be explored later.

2.1 Definitions

— In any upward drawing, the incommimg and outgoing edges of each vertex v appear consecutively in the star of v.

— S(f) = (number of sources of f) = (number of sinks of f).

— Small angles S are concidered those less than 180 degree.

— Large angles L are those greater than 180 degree.

— A pole is the source or sink of the face.

— In a planar straight line upward drawing :
  * Each pole has exactly one large angle incident on it.
  * Each internal face f has S(f) - 1 large internal angles.
  * The external face h has S(h) + 1 large internal angles.

2.2 Assignment Model

Now that we are able to find the number of large angles for each face, an assignment of poles to faces may exist such that:

— Each pole v is assigned to exactly one face f incident on v.

— Each internal face f has S(f) - 1 poles assigned to it.

— The external face h has S(h) + 1 poles assigned to it.

A consistent assignment of poles to faces can be found by using flow algorithm successively which takes $O(n^2)$ time. The procedure would be to find the maximum flow from vertices(sources and sinks) to faces. Then asssign a pole to a face such that there exists the maximum flow from the pole to face. In this computation the supply of each pole is 1, and demand of of each face is S(f) or S(h).

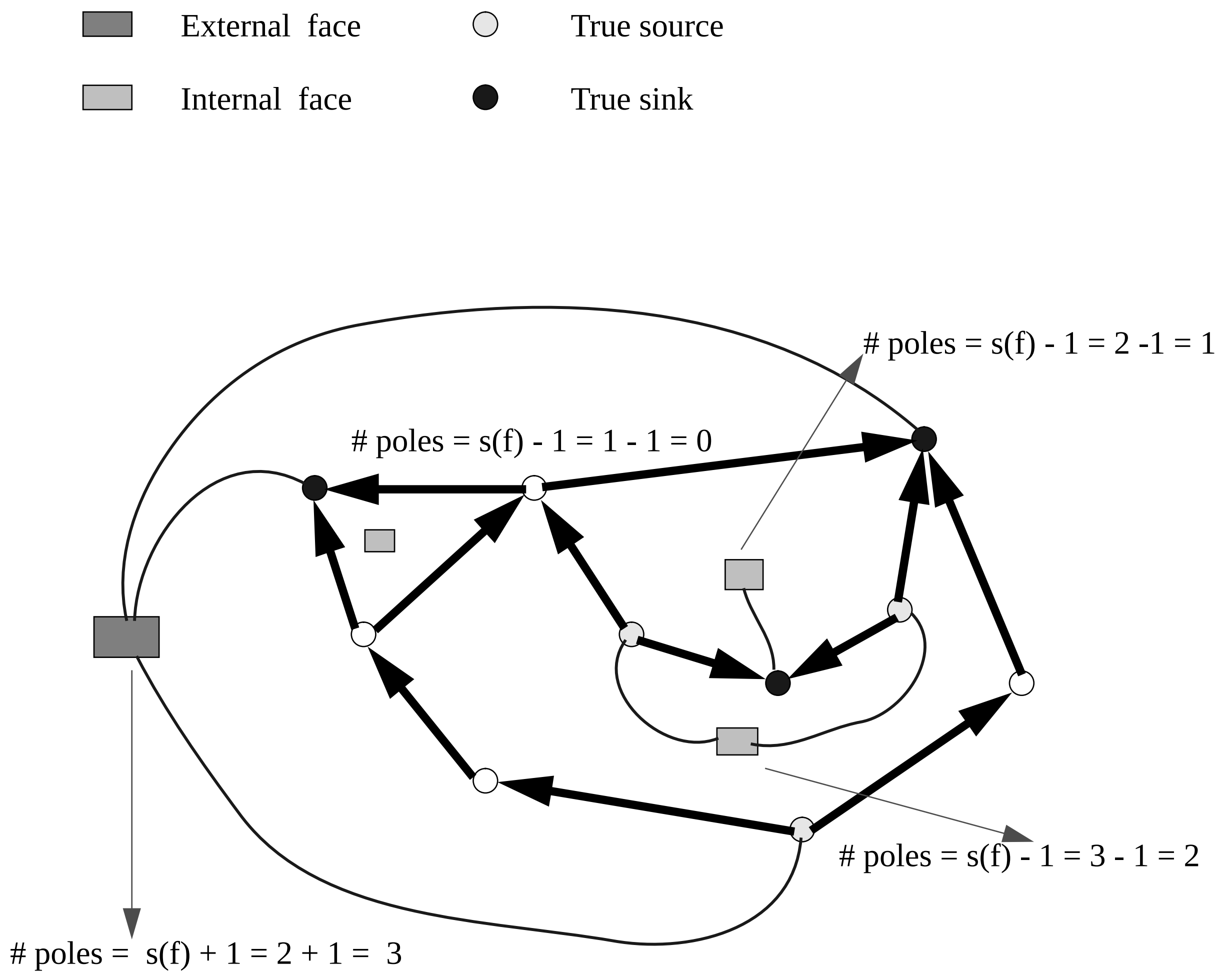

Figure 3: Assignment of poles to faces using 'flow' algorithm.

2.3 Theorem

An embedded digraph admits an upward drawing if and only if it admits a consistent assignment of poles to faces.

Proof:

If a consistent assignment of poles to faces exists (described in Assignment model section), then the poles assigned to each face are drawn with large angle in that face, and the digraph would become upward.

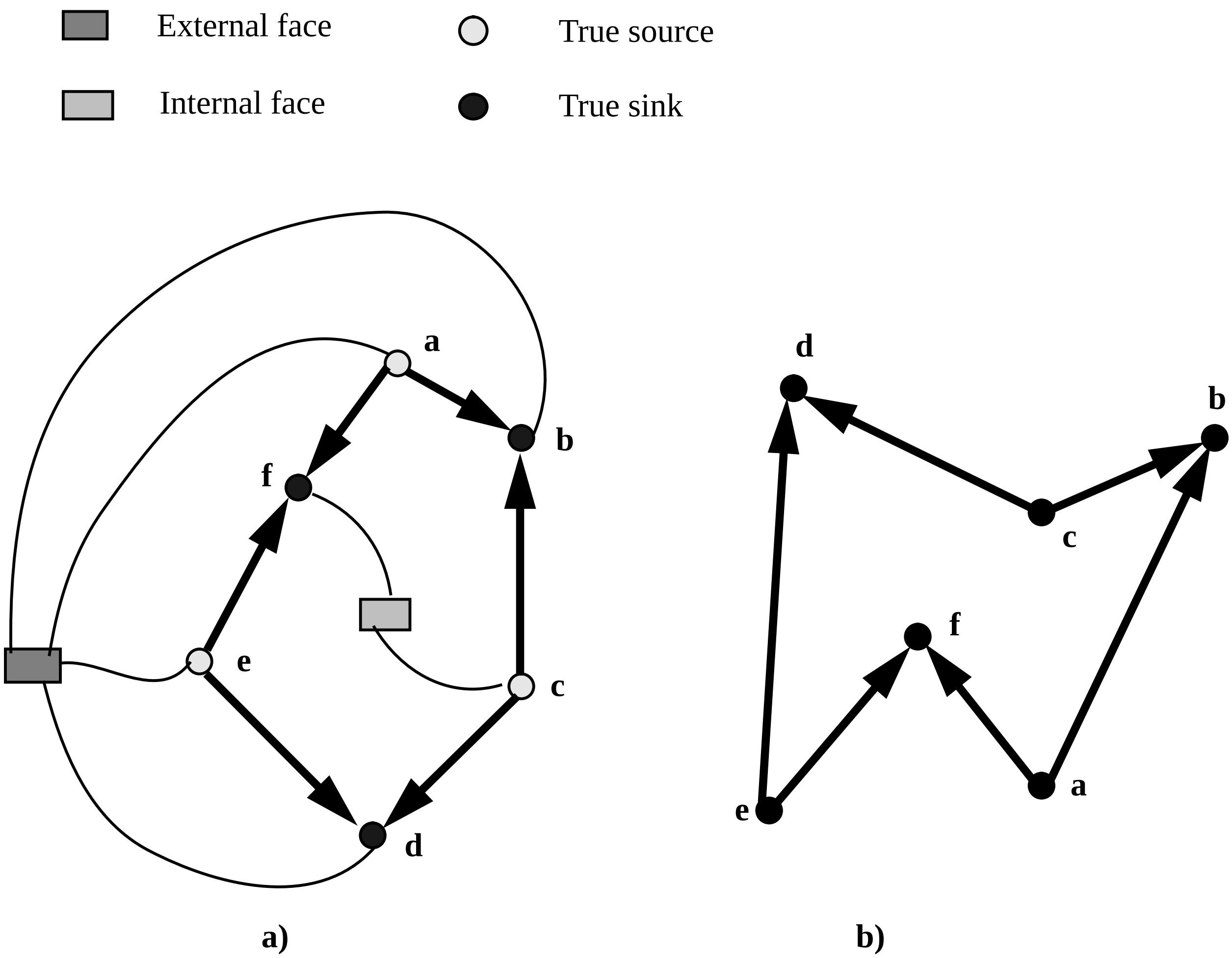

Figure 4: a) The asignment of poles to faces, b) The resulted upward drawing.

2.4 Another example:

Then again NOT every planar DAG admits an upward planar drawing. In the following example a consistent assignment of poles can not be found, since the external face needs two poles, but the graph only has one pole to offer. Therefore this DAG can not be drawn upward.

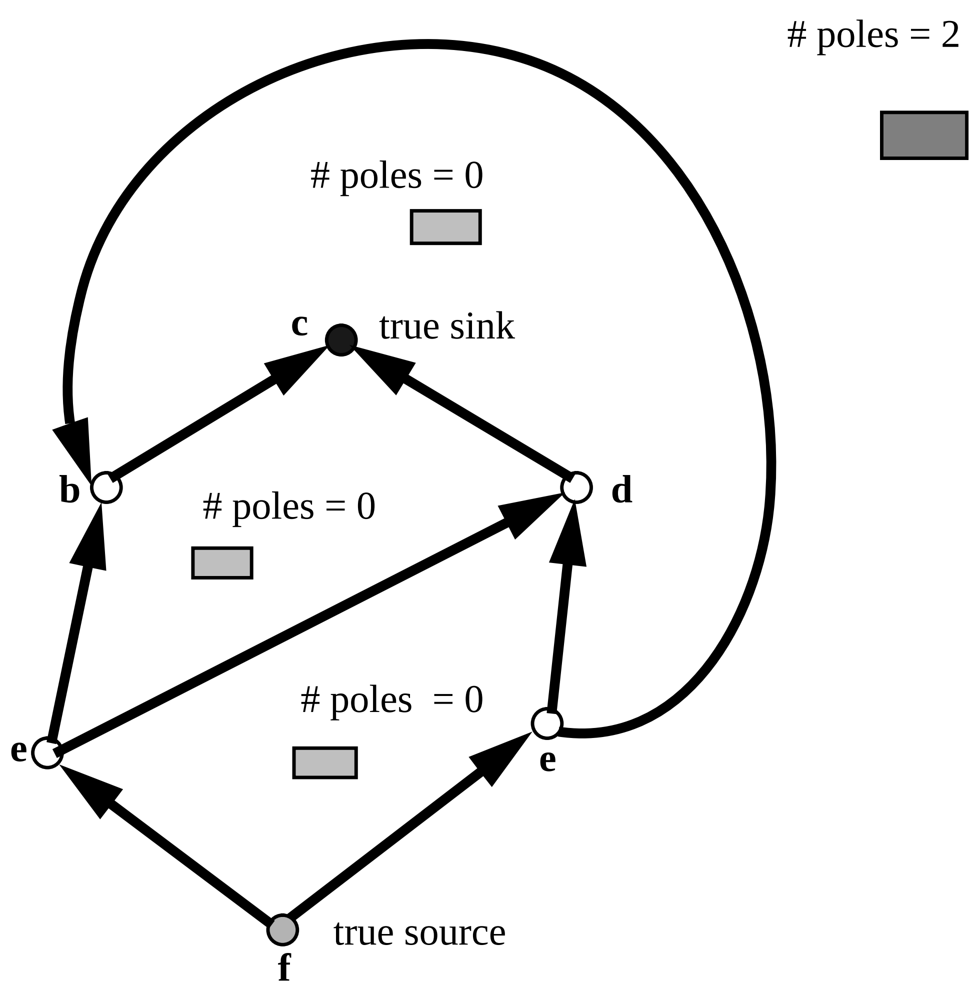

Figure 5: No upward drawing for this graph.