Introduction to Voronoi Diagrams

Lecture 13
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1 Introduction

This lecture introduces the Voronoi diagram, a general solution to 2D proximity problems. A sample of the problems addressed by this technique include Closest Pair, All Nearest Neighbors, Euclidean Minimum Spanning Tree, Triangulation and Nearest Neighbor Search (see chapter 5 of the text for this course [1] for a detailed discussion).

In general, the problem we are trying to solve is the following: Given a set S of n points in the plane, we wish to associate with each point s a region consisting of all points in the plane closer to s than any other point s' in S. This can be described formally as

$$\mathbf{Vor}(s) = \{p : distance(s, p) \leq distance(s', p), \forall s' \in S\}$$

where $\mathbf{Vor}(S)$ is the Voronoi region for a point s.

2 Voronoi Diagrams for Simple Cases

Let us first consider the simplest case for a Voronoi diagram, where S consists of a single point. In this case the Voronoi region for this point is the entire plane. Next, consider a set of two points (Figure 1a). The Voronoi diagram for the set $S = \{\mathbf{s}_1, \mathbf{s}_2\}$ consists of two half-planes divided by the ray l, which is the perpendicular bisector of $\overline{s_1 s_2}$. Note that the two regions are not disjoint, but overlap at the set of points equidistant from both points on the ray l.

Theorem 1 *All points on the half plane containing s_1 and delimited by the perpendicular bisector l of $\overline{s_1 s_2}$ are closer to s_1 than s_2.*

Proof: Consider a point p in the half-plane containing s_1 (Figure 1b). We can construct two right triangles $\triangle s_1 pb$ and $\triangle s_2 pb$. They share side $\overline{pb}$, and $\overline{s_1 b}$ is shorter than $\overline{bs_2}$ since $\|s_1 m\| = \|s_1 b\| + \|bm\|$ and $\|s_1 m\| = \|s_2 m\|$. The hypotenuse of $\triangle s_1 pb$, $\overline{s_1 p}$ is shorter than $\overline{s_2 p}$ by the Pythagorean theorem. Therefore p is closer to s_1 than s_2. □

Figure 2 shows a Voronoi diagram for three points, and the geometry used in its construction. We start by joining each pair of vertices by a line. We then draw the perpendicular bisectors to each of these lines. These three bisectors must intersect, since any three points in the plane define a circle. We then remove the portions of each line beyond the intersection and the diagram is complete. The point where the three rays intersect belongs to the Voronoi regions for all three points. This point is also the center of the circle as we prove below.

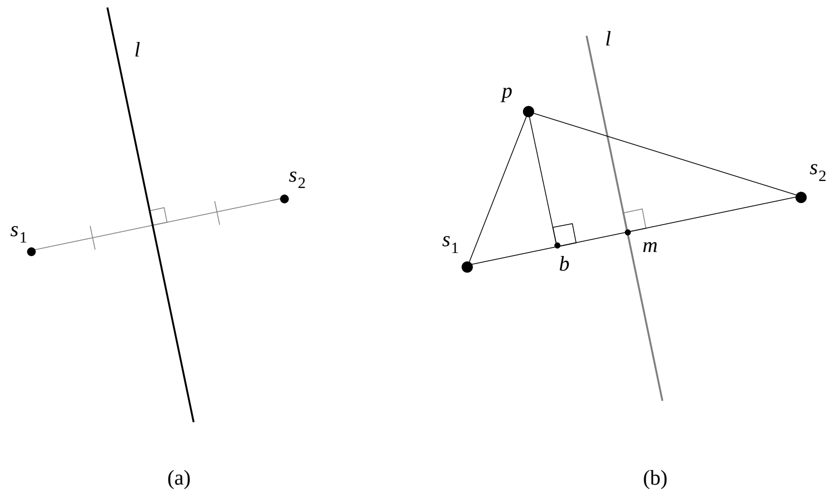

Figure 1: A Voronoi diagram for a set of two points, $S = \{s1, s2\}$. On the right a proof of its validity.

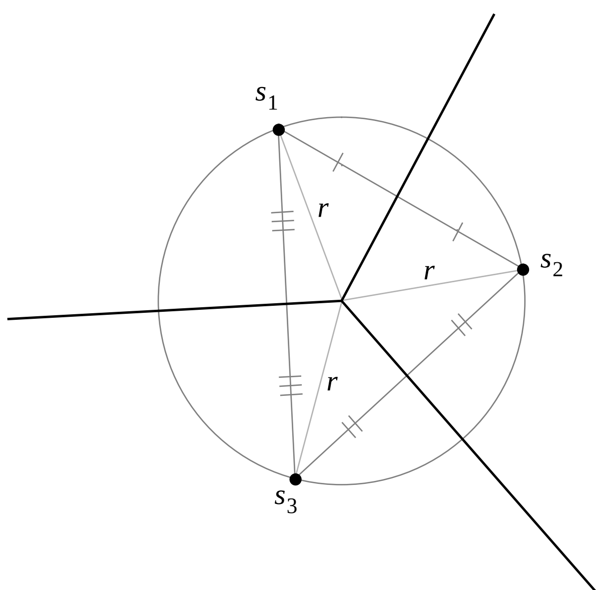

Figure 2: A Voronoi diagram for a set of three points, $S = \{s1, s2, s3\}$.

Theorem 2 *The intersection of the 3 perpendicular bisectors of s_1, s_2 and s_3 is the center of the circle containing s_1, s_2 and s_3.*

Proof: Any point on the perpendicular bisector is equidistant from the two points it bisects. Therefore segments $\overline{s_1b}$ and $\overline{s_2b}$ are equal (Figure 3). Angles $\angle s_1bc$ and $\angle s_2bc$ are right angles and both triangles share the side $\overline{bc}$. Therefore the two triangles must be congruent and their hypotenuses r_1 and r_2 are equal. A similar argument can be made between s_3 and either of s_1 or s_2, therefore point c is equidistant from s_1, s_2 and s_3 and is thus the center of the circle containing s_1, s_2 and s_3. $\square$

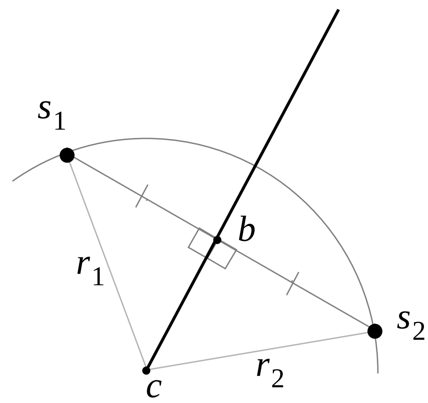

Figure 3: Proof that c is the center of the circle containing s_1, s_2 and s_3.

3 Voronoi Regions

There are several intuitive methods to construct a Voronoi region for a given point s in set S. First, we can take all of the perpendicular bisectors of the segments connecting s to the remaining members of S. We can then use these rays to delimit half-planes. The intersection of all half planes containing s is the Voronoi region for s. Or, we can start with the segments connecting s to all remaining members of S. We then gradually extend lines outward along the perpendicular bisector of these segments until they intersect (Figure 4a). Note that the points which do not contribute to the region are not necessarily the furthest away, as in Figure 4b.

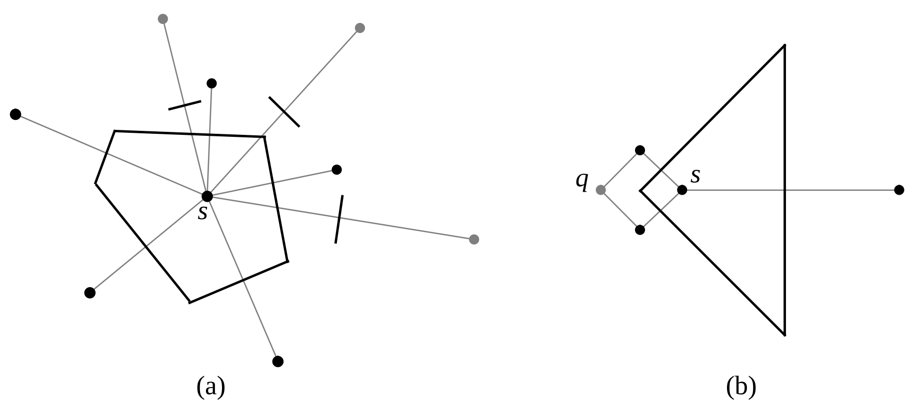

Figure 4: Constructing a Voronoi region by extending the perpendicular bisectors (a). The grey points do not contribute to the region for point s. In (b) we show that points which do not contribute to the Voronoi region for s are not necessarily the furthest away.

4 A Complete Voronoi Diagram

A Voronoi diagram is the union of all the Voronoi regions in the set:

$$\mathbf{Vor}(S) = \bigcup_{s \in S} \mathbf{Vor}(s)$$

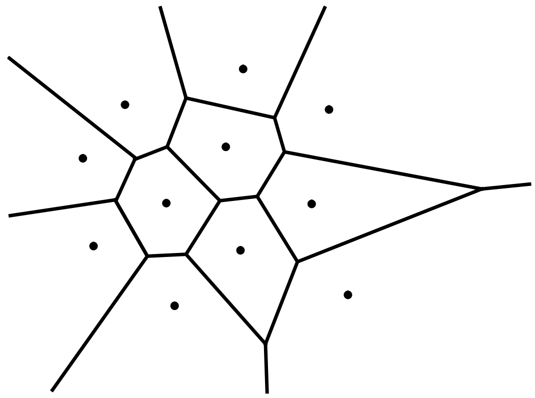

Figure 5: Voronoi diagram for a set of 11 points.

The diagram can be constructed "by hand" with the method described above for constructing each **Vor**(s). An example of a completed Voronoi diagram for a set of 11 points is shown in Figure 5.

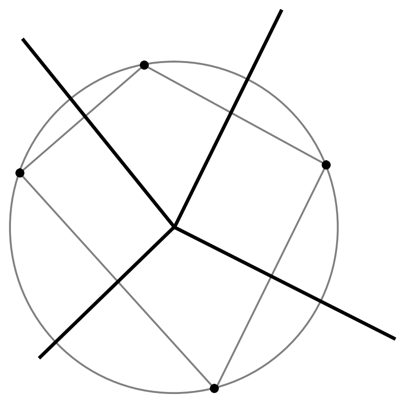

Figure 6: Four co-circular points produce a degenerate Voronoi vertex.

5 Properties of Voronoi Diagrams

Several properties now emerge. As noted before, each vertex of the Voronoi diagram is the center of a circle going through exactly three points. However, this condition is true only if no set of four co-circular points is allowed. The degenerate case can be accounted for, however the definition of Voronoi diagrams presented here disallows this case by assumption. The result of four co-circular points is illustrated in the degenerate case of Figure 6. Four lines must meet at the Voronoi vertex for these points.

Given that the degenerate case above is disallowed, we can prove the following:

Theorem 3 *The circle containing Voronoi vertex v and passing through the three points s_1, s_2 and s_3 is empty.*

Proof: Let s_1, s_2 and s_3 be the three points of S corresponding to the Voronoi vertex v (Figure 7). If $C(v)$ contains another point s_4 then s_4 must be nearer to v than any of s_1, s_2

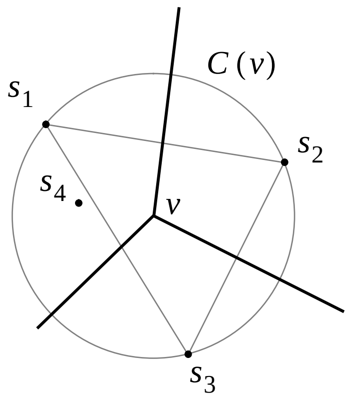

Figure 7: The circle $C(v)$ contains no other point of S.

or s_3. In this case v must be contained by the Voronoi region for s_4 and not be contained by any of the regions for s_1, s_2 or s_3, by the definition of a Voronoi region. However this is a contradiction since v is in fact common to the Voronoi regions for s_1, s_2 and s_3. □

If the points of the set S are connected through each edges of their Voronoi region, a planar graph emerges which is the dual of the voronoi diagram (Figure 8). This is also a triangulation of the set of points, referred to as the Delauney triangulation. Note that the edge of the dual may cross two edges of the Voronoi diagram, as in the rightmost edge of the dual below.

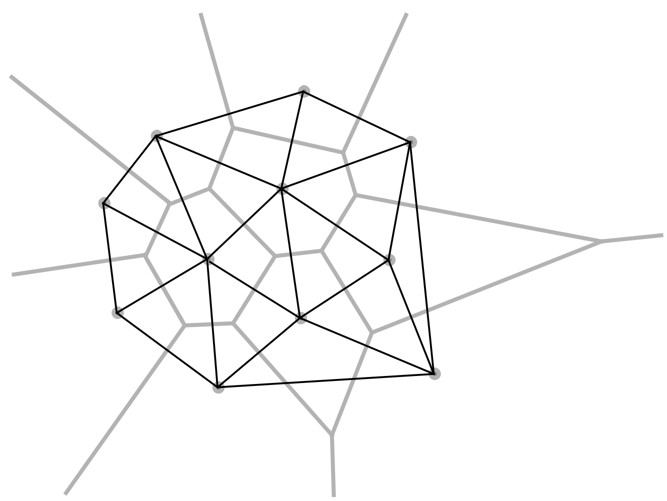

Figure 8: The dual of the Voronoi diagram.

Theorem 4 *The graph constructed by connecting all vertices in a set S across the edges of their Voronoi polygons is a triangulation of S.*

Proof: To prove that the dual is a triangulation, we must prove that each point within the dual belongs to exactly one triangle associated with a Voronoi vertex. Again, consider Figure 7. The triangle $\triangle s_1 s_2 s_3$ is part of the Delauney triangulation. No other vertex can lie within $\triangle s_1 s_2 s_3$ by the stronger condition of theorem 3 which guarantees that no other vertex will lie within $C(v)$. Each edge of a triangle interior to the convex hull shares its edges with adjacent triangles. If a point within the convex hull were not contained by a triangle, then, projecting a line out from that point in any direction would cross an edge which was not shared with any other triangle. This is a contradiction, therefore all points within the hull are contained by exactly one triangle of the dual. □

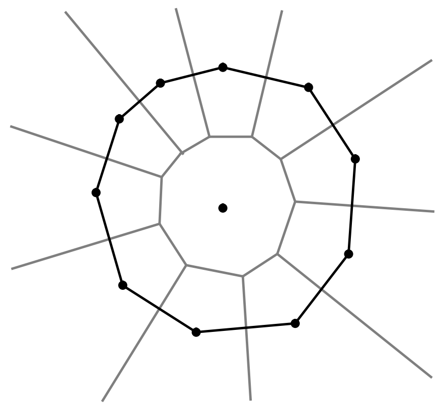

Figure 9: Unbounded regions contain the points on the convex hull of the set S.

The regions of the Voronoi diagram may be either bounded or unbounded. Every point contained in an unbounded region of the diagram lies on the convex hull of the set S. This is particularly clear in an example where all points but one lie on the convex hull (Figure 9).

6 Complexity

Figure 9 graphically illustrates the fact that a Voronoi polygon for a set of n points can have as many as $n-1$ edges. Even so, the space complexity for the *entire* Voronoi diagram is linearly bounded. Since the Voronoi diagram is a planar graph with infinite rays, we can write

$$V + R = E + 2$$

where V, R and E are the number of vertices, regions and edges respectively. Since all Voronoi vertices have 3 edges and are of degree 2, we can express E as $\frac{3}{2}V$. Thus

$$V + n = \frac{3}{2}V + 2$$

$$V = 2n - 4$$

$$V = O(n)$$

$$E = O(n)$$

We may need to add additional fictitious vertices as endpoints for the infinite rays, however the space will still be $O(n)$.

References

[1] F. Preparata, M. Shamos. *Computational Geometry.*, 204–211. Springer-Verlag, 1985.