Notes on Linear Temporal Logic

1 Syntax

For a given finite set $A$ of propositional (or Boolean) variables $p, q, \ldots$, the set of formulas of LTL (denoted $\text{LTL}(A)$) is inductively defined as the smallest set satisfying the following:

1. For every variable $p \in A$, $p \in \text{LTL}(A)$.
2. If $\alpha \in \text{LTL}(A)$ and $\beta \in \text{LTL}(A)$, then $\alpha \lor \beta \in \text{LTL}(A)$.
3. If $\alpha \in \text{LTL}(A)$ and $\beta \in \text{LTL}(A)$, then $\alpha \land \beta \in \text{LTL}(A)$.
4. If $\alpha \in \text{LTL}(A)$, then $\neg \alpha \in \text{LTL}(A)$.
5. If $\alpha \in \text{LTL}(A)$, then $\text{X} \alpha \in \text{LTL}(A)$.
6. If $\alpha \in \text{LTL}(A)$, then $\text{G} \alpha \in \text{LTL}(A)$.
7. If $\alpha \in \text{LTL}(A)$, then $\text{F} \alpha \in \text{LTL}(A)$.
8. If $\alpha, \beta \in \text{LTL}(A)$, then $\alpha \text{ U } \beta \in \text{LTL}(A)$.

2 Semantics

Instances in LTL consist of infinite sequences of the form $w = I_0, I_1, \ldots$ where each of the $I_k$ is a total function that maps each of the propositional variables in $A$ to a truth value.

For an instance $w$, we define two pieces of notation:

$$w(i) \overset{\text{def}}{=} I_i$$

The $i$-th entry in the sequence.

$$w^i \overset{\text{def}}{=} I_i, I_{i+1}, \ldots$$

The sequence starting at the $i+1$-st entry.

We now need to define the satisfaction relation for LTL. For any $\alpha \in \text{LTL}(A)$, we’ll write $w \models \alpha$ when $w$ satisfies $\alpha$, where satisfaction is defined as follows:

1. If $\alpha$ is a propositional variable $p$, then $w \models \alpha$ iff $w(0)$ maps $p$ to true.
2. If $\alpha$ is of the form $\beta \lor \gamma$, then $w \models \alpha$ iff $w \models \beta$ or $w \models \gamma$.
3. If $\alpha$ is of the form $\beta \land \gamma$, then $w \models \alpha$ iff $w \models \beta$ and $w \models \gamma$.
4. If $\alpha$ is of the form $\neg \beta$, then $w \models \alpha$ iff $w \not\models \beta$.
5. If $\alpha$ is of the form $\text{X} \beta$, then $w \models \alpha$ iff $w^1 \models \beta$.
6. If $\alpha$ is of the form $\text{G} \beta$, then $w \models \alpha$ iff for all $i \in \mathbb{N}$, $w^i \models \beta$. 
7. If \( \alpha \) is of the form \( F \beta \), then \( w \models \alpha \) iff there exists an \( i \in \mathbb{N} \) such that \( w^i \models \beta \).

8. If \( \alpha \) is of the form \( \beta \cup \gamma \), then \( w \models \alpha \) iff there exists an \( i \in \mathbb{N} \) such that \( w^i \models \gamma \) and for all \( 0 \leq j < i \), \( w^j \models \beta \).